109 Spring 2011 - Implication and Subsets - Solutions

Exercise (I.4)**.** Prove the following statements concerning positive integers a, b, and c.

(i) (a divides b) and (a divides c) $\implies$ a divides $(b + c)$.

By definition of division, the assumption says that there are integers m, n such that

$$b = ma \qquad \text{and} \qquad c = na.$$

Then $b + c = ma + na = a(m + n)$. Since $m + n$ is an integer, this means that a divides $b + c$, as required.

(ii) (a divides b) or (a divides c) $\implies$ a divides bc.

We recall that a proposition of the form $(P_1 \text{ or } P_2) \implies Q$ is equivalent to the proposition $(P_1 \implies Q)$ and $(P_2 \implies Q)$. Thus, we prove the two implications. First, we assume that a divides c. So by definition (as above), there is an integer m such that $c = ma$. Then $bc = b(ma) = a(bm)$ and a divides bc as required. To prove the second implication, we assume that b divides c. Analogously to before, there is an integer n such that $b = na$ and hence $bc = a(cn)$. Thus, a divides bc and the second implication is true. Since we have prove that both implications are true, their conjunction is also true.

Exercise (I.5)**.** Which of the following conditions are *necessary* for the positive integer n to be divisible by 6? Which are sufficient?

(i) 3 divides n
(ii) 9 divides n
(iii) 12 divides n
(iv) $n = 12$

Necessary: (i).

Sufficient: (iii), (iv).

Exercise (II.10)**.** We define half-infinite intervals as follows

$$(a, \infty) = \{x \in \mathbb{R} \mid x > a\};$$
$$[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}.$$

Prove that

(i) $(a, \infty) \subseteq [b, \infty) \iff a \geq b,$

We prove the two implications.

$\Rightarrow$ We prove the contrapositive: $a < b \implies (a, \infty) \nsubseteq [b, \infty)$. Suppose $a < b$ and consider the number $c = a + \frac{b-a}{2}$. Then

$$a < c < b.$$

In particular, by definition of the intervals, $c \in (a, \infty)$ but $c \notin [b, \infty)$. Therefore, c is a witness to the fact that $(a, \infty) \nsubseteq [b, \infty)$.

$\Leftarrow$ Suppose that $a \geq b$. To show that $(a, \infty) \subseteq [b, \infty)$ we need to show that for all $x \in \mathbb{R}$, if $x \in (a, \infty)$ then $x \in [b, \infty)$. So let x be such that $x \in (a, \infty)$. By definition, this means $x > a$. But, $a \geq b$ so transitivity gives that $x \geq b$. Thus, by definition, $x \in [b, \infty)$.

(ii) $[a, \infty) \subseteq (b, \infty) \iff a > b,$

Similar to above.