109 Spring 2011 - Injections and Surjections

Exercise (II.12)**.** Suppose that $A \subseteq \mathbb{Z}$. Write the following statement entirely in symbols using the quantifiers $\forall$ and $\exists$. Write out the negative of this statement in symbols.

There is a greatest number in the set A.

Give an example of a set A for which this statement is true. Give an example of a set A for which it is false.

$$\exists a \in A \forall x \in A (x \leq a)$$

If $A = \{1, 2, 5\}$ then the statement is true. If $A = \mathbb{Z}$ then the statement is false.

Exercise (II.17)**.** Functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined as follows.

$$f(x) = \begin{cases} x + 2 & \text{if } x < -1, \\ -x & \text{if } -1 \leq x \leq 1 \\ x - 2 & \text{if } x > 1; \end{cases} \qquad g(x) = \begin{cases} x - 2 & \text{if } x < -1, \\ -x & \text{if } -1 \leq x \leq 1 \\ x + 2 & \text{if } x > 1 \end{cases}$$

Find the functions $f \circ g$, $g \circ f$. Is g the inverse of f? Is f injective or surjective? Is g?

$$f \circ g(x) = x; \qquad g \circ f(x) = \begin{cases} x & \text{if } x < -3, \\ -(x + 2) & \text{if } -3 \leq x < -1 \\ x & \text{if } -1 \leq x \leq 1 \\ -(x - 2) & \text{if } 1 < x \leq 3 \\ x & \text{if } x > 3 \end{cases}$$

Since $g \circ f$ is not the identity map, g is not the inverse of f.

f is not injective. For a counterexample, take $x_1 = -2, x_2 = 2$. Then

$$f(x_1) = 0 = f(x_1) \qquad \text{but} \qquad x_1 \neq x_2.$$

f is surjective. Let $y \in \mathbb{R}$. Then if $y > 1$, consider $x = y + 2$. Since $x > 3 > 1$, $f(x) = x - 2 = y$. On the other hand, if $y \leq 1$, consider $x = y - 2$. Since $y \leq 1$, $x \leq -1$ so $f(x) = x + 2 = y$.

g is not surjective. For a counterexample, take $y = -2$. If y is the image of some $x < -1$, it must be that $-2 = f(x) = x - 2$, hence $x = 0$ which is not in the appropriate interval. If y is the image of some $-1 \leq x \leq 1$, it must be that $-2 = f(x) = -x$ so $x = 2$, again not in the interval. Finally, if y is the image of some $x > 1$, then $-2 = f(x) = x + 2$ so $x = -4$, not in the interval. Thus, $y = -2$ is not the image of any $x \in \mathbb{R}$ and g is not surjective.

g is injective. Suppose $y = g(x_1) = g(x_2)$. The image of g can be split into three disjoint pieces: $y < -3$, $-1 \leq y \leq 1$, $y > 3$. Each of these corresponds to an interval in the domain. The functions $x - 2, -x, x + 2$ are each injective on their respective intervals of the definition. Therefore, so is g.

Exercise (II.18).

(a) Suppose that $f : X \to Y$ and $g : Y \to Z$ are surjections. Prove that the composite $g \circ f : X \to Z$ is a surjection.

Let $z \in Z$. Since g is a surjection, there is $y \in Y$ such that $g(y) = z$. Since f is a surjection, there is $x \in X$ such that $y = f(x)$. Then

$$(g \circ f)(x) = g(f(x)) = g(y) = z.$$

(b) Suppose that $f : X \to Y$ and $g : Y \to Z$ are injections. Prove that the composite $g \circ f : X \to Z$ is an injection.

Suppose $g \circ f(x_1) = g \circ f(x_2)$. Then $g(f(x_1)) = g(f(x_2))$. Since g is an injection, this means that $f(x_1) = f(x_2)$. But, f is also an injection so $x_1 = x_2$.