# 109 Spring 2011 - Limits and Functions - Solutions

**Exercise.** We say that a sequence $f : \mathbb{Z}^+ \to \mathbb{R}$ is **bounded** if there is some number $b \in \mathbb{R}$ such that for each $n \in \mathbb{Z}^+$,

$$|f(n)| \leq b.$$

(a) Prove that the harmonic sequence is bounded.

Let $b = 1$. Then for each $n \in Z^+$

$$f(n) = \frac{1}{n} \leq 1 = b.$$

(b) Prove that the sequence of (positive) powers of 2 is not bounded.

We need to prove that for each $b \in \mathbb{R}$, there is some $n \in Z^+$ such that $f(n) > b$. So let $b$ be an arbitrary real number. If $b \leq 0$, each positive power of 2 is greater than it. So, suppose $b > 0$. Let $n_b$ be the smallest integer bigger than or equal to $b$, $n_b = \lceil b \rceil$. Then

$$f(n_b) = 2^{n_b} \geq 2^n > n.$$

**Exercise** (Hard!)**.** Prove that if a sequence converges to a finite limit then it is bounded.

*To make the proof more readable, we prove below the (weaker) statement that if a sequence has positive values (that is, $f : \mathbb{Z}^+ \to \mathbb{R}^+$) and converges to a finite limit then it is bounded. The proof of the general case is very similar but a little longer.*

*Proof.* Let $f : \mathbb{Z}^+ \to \mathbb{R}^+$ be a convergent sequence, say with limit $L$. By definition, this means that

$$\forall \epsilon \in \mathbb{R}^+ \exists N \in \mathbb{Z}^+ \forall n \in \mathbb{Z}^+ \left( n \geq N \implies |f(n) - L| < \epsilon \right).$$

In particular, consider $\epsilon = 1$. Then we are guaranteed the existence of some number, call it $N_1$ such that

$$\forall n \in \mathbb{Z}^+ (n \geq N_1 \implies |f(n) - L| < 1).$$

That is, for each $n \geq N_1$,

$$L - 1 \leq f(n) \leq L + 1.$$

We define

$$B = \max\{f(1), f(2), \ldots, f(N_1), L + 1\}$$

and claim that $B$ is a bound on the sequence $f$. To prove this claim, consider $n \in \mathbb{Z}^+$. Then

- if $1 \leq n \leq N_1$, $|f(n)| = f(n) \leq \max\{f(1), f(2), \ldots, f(N_1)\} \leq B$;
- otherwise, $n > N_1$ so $|f(n)| = f(n) \leq L + 1 \leq B$.

Thus, for each $n$, $|f(n)| < B$ as required. □

The converse is not true. As a counterexample, we can consider the sequence

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

**Exercise** (II.16)**.** Determine which of the following functions $f_i : \mathbb{R} \to \mathbb{R}$ are injective, which are surjective, and which are bijective. Write down an inverse function of each of the bijections.

(i) $f_1(x) = x - 1$;
Bijection. The inverse is $f_1^{-1}(x) = x + 1$.

(ii) $f_2(x) = x^3$;
Bijection. The inverse is $f_2^{-1}(x) = x^{1/3}$.

(iii) $f_3(x) = x^3 - x$;
Surjective but not injective.

*Proof.* To prove that $f_3$ is surjective, we use the graph of the function. One can draw the graph and observe that every altitude is achieved. To make this precise, one could use calculus to find local maxima / minima and apply the Intermediate Value Theorem to find preimages of each given $y$ value. □

*Proof.* To prove that $f_3$ is not injective, we find $x_1 \neq x_2$ such that $f_3(x_1) = f_3(x_2)$. For example, we can choose $x_1 = 0$ and $x_2 = 1$. Then

$$f_3(x_1) = 0 - 0 = 0 = 1^3 - 1 = f_3(x_2).$$

□

(iv) $f_4(x) = x^3 - 3x^2 + 3x - 1$;
Bijection. We can rewrite this as $f_4(x) = (x - 1)^3$. Therefore, its inverse is

$$f_4^{-1}(x) = x^{1/3} + 1.$$

(v) $f_5(x) = e^x$;
Injective but not surjective.

*Proof.* To prove that $f_5$ is injective we prove the universal implication $\forall x_1 \forall x_2 (f(x_1) = f(x_2) \implies x_1 = x_2)$. Let $x_1, x_2 \in \mathbb{R}$ and suppose $f(x_1) = f(x_2)$. That is, $e^{x_1} = e^{x_1}$. Applying the natural logarithm and using logarithm rules, we get

$$x_1 = \ln(e^{x_1}) = \ln(e^{x_2}) = x^2,$$

as required. □

*Proof.* To prove that $f_5$ is not surjective, we find $y \in \mathbb{R}$ such that for no $x \in \mathbb{R}$ is it the case that $e^x = y$. Any nonpositive number will work since $e^x > 0$ for all $x$. □

(vi) $f_6(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x \leq -0 \end{cases}$
Bijection. The inverse is

$$f_6^{-1}(x) = \begin{cases} -\sqrt{-x} & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$$