MATH 109 - Sample Long-Form Solution

Statement of problem

Question. Do there exist propositions p and q such that both "p implies q" and its converse are true? Do there exist propositions p and q such that both "p implies q" and its converse are false?

def

Before answering the question, we recall some key definitions. The proposition "p implies q", abbreviated as $p \implies q$, is defined by the following truth-table (see p. 11 of Eccles).

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

references

def.

The converse of the implication $p \implies q$ is defined as $q \implies p$ (see p. 14 of Eccles).

First, we consider whether there can be propositions p and q such that both $p \implies q$ and $q \implies p$ are true. If p and q are both true propositions then both $p \implies q$ and $q \implies p$ are true by the first line of the truth-table. Similarly, if p and q are both false then both $p \implies q$ and $q \implies p$ are true by the last line of the truth-table. So, one example of propositions p and q such that both "p implies q" and its converse are true is

guiding text

$$p: \text{`}0 < 1\text{'} \qquad \text{and} \qquad q: \text{`}2 \neq 3\text{'}.$$

Another example is

$$p: \text{`}\pi < e\text{'} \qquad \text{and} \qquad q: \text{`}0 = 1\text{'}.$$

The second part of the question asks for propositions where the implication and its converse are both false. By inspection of the truth-table, this seems less likely to be possible since there is only one combination of truth values of p and q which makes $p \implies q$ false. We now prove that it is, in fact, impossible to find such propositions.

guiding text

Proof. By contradiction, suppose that there were propositions p and q such that $p \implies q$ is false and $q \implies p$ is also false. Since $p \implies q$ is false, it must be the case that p is true and q is false (by the third line of the truth-table). Similarly, since $q \implies p$ is false, q is true and p is false. Thus, p must be both true and false, which is impossible. We arrived at contradiction when we assumed that there were such propositions so we conclude that there are no propositions where the implication and its converse are both false. □