MATH 109	Assignment 3
Due: 4/18/11	

#7.2 (i) **Counterexample:** This is the statement that if $m \in \mathbb{Z}^+$ then

$$\{n \in \mathbb{Z}^+ : m \leq n\} = \mathbb{Z}^+.$$

When $m = 2$, we have $1 \notin \{n \in \mathbb{Z}^+ : 2 \leq n\}$ so the set does not equal $\mathbb{Z}^+$. $\square$

(iii) **Proof.** This is the statement that if $m \in \mathbb{Z}^+$ then the set $\{n \in \mathbb{Z}^+ : m \leq n\}$ is nonempty. Since $m \in \{n \in \mathbb{Z}^+ : m \leq n\}$, we see that the statement is true for all integers $m \in \mathbb{Z}^+$. $\square$

(v) **Proof.** This is the statement that if $n \in \mathbb{Z}^+$ then the set $\{m \in \mathbb{Z}^+ : m \leq n\}$ is nonempty. Since $n \in \{m \in \mathbb{Z}^+ : m \leq n\}$, we see that the statement is true for all integers $n \in \mathbb{Z}^+$. $\square$

#7.4 (iii) **Proof.** This is the statement that if $x \in \mathbb{R}$ then the set $\{y \in \mathbb{R} : xy = 0\}$ is nonempty. Since $x \cdot 0 = 0$, we see that $0 \in \{y \in \mathbb{R} : xy = 0\}$, and hence the set is nonempty. $\square$

(iv) **Proof.** This is the statement that the set $\{y \in \mathbb{R} : \forall x \in \mathbb{R}, xy = 0\}$ is nonempty. Since $x \cdot 0 = 0$, we see that $0 \in \{y \in \mathbb{R} : \forall x \in \mathbb{R}, xy = 0\}$, and hence the set is nonempty. $\square$

#8.1 **Proof.** To show that g is well-defined, we need to show that to each $(x, y) \in \mathbb{R}^2$, the function g assigns a unique real number. Let $(x, y) \in \mathbb{R}$. By trichotomy (Axiom 3.1.2 (i)), we have that exactly one of the three possibilities $x < y,\ x = y,\ x > y$ is true.

Case 1: If $x < y$, then $x \leq y$ and $x \ngeq y$ so $g(x, y) = y$ is well-defined.

Case 2: If $x = y$ then $x \leq y$ so $g(x, y) = x$. We also have, however, that $x \geq y$ so $g(x, y) = y$. Since $x = y$, we see that g is well-defined in this case because $g(x, y) = x = y$.

Case 3: If $x > y$, then $x \geq y$ and $x \nleq y$ so $g(x, y) = x$ is well-defined. This concludes the proof that g is well-defined.

To show that $g = f$, we must show that for each $(x, y) \in \mathbb{R}^2$ we have $g(x, y) = f(x, y)$. Let $(x, y) \in \mathbb{R}$. By trichotomy (Axiom 3.1.2 (i)), we have that exactly one of the three possibilities $x < y,\ x = y,\ x > y$ is true.

Case 1: If $x < y$, then $|x - y| = y - x$ so

$$f(x, y) = \frac{x + y}{2} + \frac{|x - y|}{2} = \frac{x + y}{2} + \frac{y - x}{2} = y = g(x, y)$$

Case 2: If $x = y$ then

$$f(x, y) = \frac{x + y}{2} + \frac{|x - y|}{2} = \frac{x + y}{2} + 0 = \frac{x + y}{2} = \frac{x + x}{2} = x = g(x, y).$$

Case 3: If $x > y$, then $|x - y| = x - y$ so

$$f(x, y) = \frac{x + y}{2} + \frac{|x - y|}{2} = \frac{x + y}{2} + \frac{x - y}{2} = x = g(x, y).$$

Hence, we have shown that $g = f$. □

#8.4 **Proof.** We are required to prove that for all $\epsilon \in \mathbb{R}^+$, there exists $N \in \mathbb{Z}^+$ such that for all $n \in \mathbb{Z}^+$ with $n \geq N$ we have $1/n < \epsilon$. Given a positive real $\epsilon \in \mathbb{R}^+$, we have $1/n < \epsilon$ if and only if $n > 1/\epsilon$. Hence, if we choose $N > 1/\epsilon$ then $n \geq N > 1/\epsilon$ implies $1/n < \epsilon$ as desired. □

Continuity(a): **Proof.** We are required to show that for each $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in \mathbb{R}$ and $0 < |x - 0| < \delta$, then $|f(x) - 1| < \epsilon$. Given $\epsilon > 0$, let $\delta = \min\{\epsilon, 1\}$. Since $\delta \leq 1$, we have $x \neq 1$, and thus

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \;\Rightarrow\; |f(x) - 1| = |(x + 1) - 1| = |x - 0| < \min\{\epsilon, 1\} \leq \epsilon. \quad \square$$

Continuity (b): **Proof.** To show that f is not continuous at $x = 0$, we must show that there exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists an $x \in \mathbb{R}$ such that $0 < |x - 0| < \delta$ and $|f(x) - f(0)| \geq \epsilon$. Let $\epsilon = 1/2$. We will show that for all $\delta > 0$, there exists an $x \in \mathbb{R}$ such that $|x| < \delta$ and $|f(x)| > 1/2$. In part(a), we showed that $\lim_{x \to 0} = 1$. This means that there exists a $\hat{\delta} > 0$ such that for all $x \in \mathbb{R}$ with $|x| < \hat{\delta}$ we have $|f(x) - 1| < 1/4$; in other words all $x \in \mathbb{R}$ with $|x| < \hat{\delta}$ satisfy $3/4 < f(x) < 5/4$. Consequently, for any $\delta > 0$, if we choose $x \in \mathbb{R}$ such that $|x| < \min\{\delta, \hat{\delta}\}$ then $|x| < \delta$ and $|f(x)| > 1/2$. □