MATH 109	**Assignment 4**
Due: 4/25/11	

#9.1 (i) We claim that $f(x) = 2x + 5$ is both an injection and a surjection, and hence a bijection. To prove that f is an injection we have to show that $f(x) = f(y)$ implies that $x = y$. Hence, suppose that $2x+5 = f(x) = f(y) = 2y+5$. By subtracting 5 from both sides of the equation, and then dividing by 2, we see that $x = y$ as desired, so f is an injection. To prove that f is a surjection, we must show that for any $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) = y$. Given $y \in \mathbb{R}$, let $x = (y - 5)/2$. We then have

$$f(x) = f\left(\frac{y-5}{2}\right) = 2\left(\frac{y-5}{2}\right) + 5 = (y-5) + 5 = y,$$

as desired. Hence, f is both an injection and a surjection, and hence a bijection. $\square$

(ii) We claim that f is neither an injection nor a surjection, and hence not a bijection. To see that f is not an injection, observe that $f(-2) = f(0) = 1$. To see that f is not a surjection, observe that $f(x) = (x-1)^2 \geq 0$; hence there does not exist $x \in \mathbb{R}$ for which $f(x) = -1$. This proves that f is neither an injection nor a surjection, and hence not a bijection. $\square$

#9.4 **Proof.** We are given that $f : X \to Y$ and $g : Y \to Z$ are injections, and we want to show that $g \circ f : X \to Z$ is also an injection. Suppose that $g \circ f(x) = g \circ f(y)$. This means that $g(f(x)) = g \circ f(x) = g \circ f(y) = g(f(y))$. As g is an injection, we have $f(x) = f(y)$, and since f is an injection, we have $x = y$. This proves that $g \circ f$ is an injection. $\square$

#10.3 **Proof.** Since $|X| = 3$ and $|Y| = 2$, by the multiplication principle (Theorem 10.2.3), we have $|X \times Y| = |X||Y| = 3 * 2 = 6$. Define $f : \mathbb{N}_6 \to X \times Y$ by

$$f(1) = (a, d),\ f(2) = (a, e),\ f(3) = (b, d),\ f(4) = (b, e),\ f(5) = (c, d),\ f(6) = (c, e).$$

We see that f is an explicit bijection between $\mathbb{N}_6$ and $X \times Y$. $\square$

#11.2 **Proof.** Suppose, for a contradiction, that $A \subset \mathbb{R}$ is a set of real numbers with two distinct maximum elements c_1 and c_2. By definition, we have $c_1 \in A$ and $c_1 \geq a$ for all $a \in A$. Similarly, $c_2 \in A$ and $c_2 \geq a$ for all $a \in A$. In particular, we have $c_1 \geq c_2$ and $c_2 \geq c_1$. By trichotomy (Axiom 3.1.2 (i)), exactly one of the three possibilities $c_1 < c_2$, $c_1 = c_2$, $c_1 > c_2$ holds. Hence $c_1 = c_2$, which contradicts our original assumption that c_1 and c_2 are distinct. This proves that if a set $A \subset \mathbb{R}$ of real numbers has a maximum element, then this element is unique. $\square$

#III.2 **Solution.** Let X, Y, Z be the sets of students who liked Reasoning, Algebra, and Calculus respectively. By Proposition 10.3.2, we have

$$|X \cap Y \cap Z| = |X \cup Y \cup Z| - |X| - |Y| - |Z| + |X \cap Y| + |X \cap Z| + |Y \cap Z|. \qquad (1)$$

We will determine each of the quantities on the right hand side. Note that $|X \cup Y \cup Z|$ is the number of students who liked at least one of the modules. Let U be the universe of all students who took first year core modules last year; we have $|U| = 170$. We are given that two students liked none of the modules so $|(X \cup Y \cup Z)^c| = 2$. By the addition principle (Theorem 10.2.1), we have $|X \cup Y \cup Z| = 170 - 2 = 168$.

We are given that $|X| = |Y| = |Z| = 124$. Observe that by Proposition 10.3.2, we have

$$\begin{aligned} 124 = |X| &= |X \cap Y| + |X \cap Z| + |X \setminus ((X \cap Y) \cup (X \cup Z))| - |X \cap Y \cap Z| \\ &= |X \cap Y| + |X \cap Z| + 10 - |X \cap Y \cap Z| \qquad (2) \end{aligned}$$

since 10 students like Reasoning only.

Similarly, since no students like Algebra only, we have

$$\begin{aligned} 124 = |Y| &= |X \cap Y| + |Y \cap Z| + |Y \setminus ((X \cap Y) \cup (Y \cup Z))| - |X \cap Y \cap Z| \\ &= |X \cap Y| + |Y \cap Z| - |X \cap Y \cap Z|. \qquad (3) \end{aligned}$$

Since 4 students like Calculus only, Proposition 10.3.2 yields

$$\begin{aligned} 124 = |Z| &= |X \cap Z| + |Y \cap Z| + |Z \setminus ((X \cap Z) \cup (Y \cup Z))| - |X \cap Y \cap Z| \\ &= |X \cap Z| + |Y \cap Z| + 4 - |X \cap Y \cap Z|. \qquad (4) \end{aligned}$$

Adding together (2), (3), and (4) and simplifying yields

$$358 = 2(|X \cap Y| + |X \cap Z| + |Y \cap Z|) - 3|X \cap Y \cap Z|.$$

Hence, we see that

$$|X \cap Y| + |X \cap Z| + |Y \cap Z| = 179 + \frac{3}{2}|X \cap Y \cap Z|.$$

Plugging this into (1) gives

$$|X \cap Y \cap Z| = 168 - 124 - 124 - 124 + 179 + \frac{3}{2}|X \cap Y \cap Z|.$$

Hence, $|X \cap Y \cap Z| = -2(168 - 124 - 124 - 124 + 179) = 50$. □