**MATH 109** **Assignment 5**
**Due: 5/2/11**

#10.1 **Proof.** We are given that $X = \{x_1, \ldots, x_n\}$ is a finite set of cardinality $n$. Let $Y$ be a set, and first suppose there is a bijection $f : X \to Y$. Since $f$ is injective, we have $f(x_1), \ldots, f(x_n)$ are all distinct. Hence $|Y| \geq n$. On the other hand, since $f$ is surjective, we have $Y \subset \{f(x_1), \ldots, f(x_n)\}$; hence $|Y| \leq n$. This implies that $|Y| = n$.

Now suppose that $Y = \{y_1, \ldots, y_n\}$ has cardinality $n$. It is easy to see that the function $f : X \to Y$ defined by $f(x_i) = y_i$ is a bijection. Hence, we have shown that if $X$ is a finite set of cardinality $n$, then a set $Y$ has $|Y| = n$ if and only if there exists a bijection $X \to Y$. □

#11.5(ii) **Proof.** We are given that $X$ and $Y$ are finite sets with $X \subseteq Y$. First suppose that $|X| < |Y|$. We must have some $y \in Y$ is not contained in $X$ (otherwise $|X| \geq |Y|$) so $X \neq Y$; this implies that $X \subset Y$.

Now assume that $X \subset Y$. By definition,

(i) all $x \in X$ are contained in $Y$ (so $|X| \leq |Y|$ by Corollary 11.1.5) and

(ii) there exists $\hat{y} \in Y$ such that $\hat{y} \notin X$.

By (i) and (ii), we have $X \cup \{\hat{y}\} \subset Y$. Since $X$ and $\{\hat{y}\}$ are disjoint, by the addition principle, we have

$$|X| < |X| + 1 \leq |Y|.$$

#5.2 **Proof.** We first establish that the statement is true for the base case $m = 10$. We have

$$10^3 = 1000 < 1024 = 2^{10}.$$

Now we assume the statement is true for $m$ and prove it for $m + 1$. Observe that

$$\frac{m+1}{m} \leq \frac{11}{10} \Leftrightarrow 11m \leq 10m + 10 \Leftrightarrow 10 \leq m.$$

Hence, by the induction hypothesis, we have

$$(m+1)^3 = \left(\frac{m+1}{m}\right)^3 m^3 \leq \left(\frac{m+1}{m}\right)^3 2^m \leq \left(\frac{11}{10}\right)^3 2^m \leq 2 \cdot 2^m = 2^{m+1}.$$

We have proved that $m^3 \leq 2^m$ for all integers $m \geq 10$. □

#5.4 **Proof.** We first establish that the statement is true for the base case $n = 0$. Since $x \neq 1$, we have

$$\sum_{i=0}^{0} x^i = 1 = \frac{1 - x^{0+1}}{1 - x}.$$

Now we assume the statement is true for $n$ and prove it for $n+1$. By the induction hypothesis, we have

$$\begin{aligned}\sum_{i=0}^{n} x^i = x^n + \sum_{i=0}^{n-1} x^i = x^n + \frac{1 - x^n}{1 - x} &= \frac{(1-x)x^n}{1-x} + \frac{1 - x^n}{1 - x} \\ &= \frac{x^n - x^{n+1}}{1 - x} + \frac{1 - x^n}{1 - x} = \frac{1 - x^{n+1}}{1 - x}\end{aligned}$$

We have proved that $\sum_{i=0}^{n} x^i = (1 - x^{n+1})/(1 - x)$ for any integer $n \geq 0$ and any real $x \neq 1$. $\square$

#I.13 **Proof.** We first establish the statement for the base case $n = 4$. We have

$$4! = 24 > 16 = 2^4.$$

Now we assume the statement is true for $n$ and prove it for $n + 1$. Since $n \geq 4$, by the induction hypothesis, we have $(n + 1)! = (n + 1)n! > (n + 1)2^n > 2 \cdot 2^n = 2^{n+1}$. We have proved that $n! > 2^n$ for all integers $n \geq 4$. $\square$