#12.4 Solution. Let $X, Y, Z$ be the possible dishes that each of the three diners can choose respectively. Since the menu consists of five items we have $|X| = |Y| = |Z| = 5$. The set $X \times Y \times Z$ represents the set of choices possible if we record who selected which dish. Hence, by the multiplication principle (Theorem 10.2.3), there are $|X \times Y \times Z| = |X||Y||Z| = 5^3$ choices.

If we do not care who orders which dish, then there are

(i) 5 ways for the diners to choose one dish from the menu,
(ii) $\binom{5}{2} \cdot 2 = 20$ ways for the diners to choose two dishes from the menu, since one dish is chosen once and the other dish is chosen twice,
(iii) $\binom{5}{3} = 10$ ways for the diners to choose three dishes from the menu.

Hence, in total, there are 35 choices if we do not care who orders which dish. 

#14.1 Proof. Since $A$ is finite, we have $A \setminus B$ is too, so write $A \setminus B = \{x_1, \ldots, x_n\}$. Observe that $A \cup B = (A \setminus B) \cup B$ and that $A \setminus B$ and $B$ are disjoint. As $B$ is denumerable, there is a bijection $g : \mathbb{Z}^+ \rightarrow B$. Now define a function $h : \mathbb{Z}^+ \rightarrow A \cup B$ by

$$h(i) = \begin{cases} x_i & \text{if } 1 \leq i \leq n \\ g(i-n) & \text{if } n < i. \end{cases}$$

Clearly $h$ is well-defined, and we now show that $h$ is a bijection. First, we will show that $h$ is an injection. Suppose that $h(i) = h(j)$. Since $h(i), h(j) \in A \setminus B$ for $1 \leq i, j \leq n$ and $h(i) \in B$ for $n < i$, we must have either $1 \leq i, j \leq n$ or $n < i, j$ because $A \setminus B$ and $B$ are disjoint. In the first case, we see $h(i) = h(j)$ implies $i = j$ since $x_1, \ldots, x_n$ are distinct. In the second case, we see $h(i) = h(j)$ implies that $g(i-n) = h(i) = h(j) = g(j-n)$ so $i-n = j-n$ as $g$ is an injection; this implies $i = j$ as desired.

Now we will show that $h$ is a surjection. If $w \in A \cup B$, then $w \in A \setminus B$ or $w \in B$. If $w \in A \setminus B$, then $w = x_i$ for some $1 \leq i \leq n$ so $h(i) = w$. If $w \in B$, then there is $i \in \mathbb{Z}^+$ such that $g(i) = w$ since $g$ is a surjection. Hence $h(i+n) = g(i) = w$. This shows that $h$ is a surjection. We have proved that $h : \mathbb{Z}^+ \rightarrow A \cup B$ is a bijection and so $A \cup B$ is denumerable. 

#14.4 Proof. Suppose that $X$ and $Y$ are two sets such that $|X| \leq |Y|$ and $|Y| \leq |X|$. We prove that $|X| = |Y|$ by contradiction. Suppose, for a contradiction, that $|X| \neq |Y|$. Then since $|X| \leq |Y|$, we have $|X| < |Y|$, and since $|Y| \leq |X|$, we have $|Y| < |X|$. By definition,
if \(|X| < |Y|\), then there exists an injection \(X \to Y\). By the Cantor-Schroder-Bernstein theorem, however, if \(|Y| < |X|\), there does not exist an injection \(X \to Y\). Thus we have obtained a contradiction to our assumption that \(|X| \neq |Y|\) is false. Hence, \(|X| = |Y|\) as required.

#15.1 (i) We have 100 = 3(33) + 1 so \(q = 33\) and \(r = 1\).

(ii) We have 3 = 100(0) + 3 so \(q = 0\) and \(r = 3\).

#15.2 Proof. First suppose that 5 \(| a \). We will show that 5 \(| a^2 \). Let \(x \in \mathbb{Z}\) be the integer such that \(a = 5x\). We have \(a^2 = (5x)^2 = 5(5x^2)\) so 5 \(| a^2 \).

Now suppose 5 \(| a^2 \). We will show that 5 \(| a \). By the division theorem (Theorem 15.1.1), there are unique integers \(q, r \in \mathbb{Z}\) such that \(a = 5q + r\) and \(0 \leq r < 5\). We then have

\[
a^2 = (5q + r)^2 = 25q^2 + 10qr + r^2 = 5(5q^2 + 2qr) + r^2.
\]

Since 5 \(| a^2 \), it follows that 5 \(| r^2 \). As \(r \in \{0, 1, 2, 3, 4\}\), we must have that \(r = 0\). Hence 5 \(| a \). We have shown that an integer \(a\) is divisible by 5 if and only if \(a^2\) is divisible by 5.