MATH 109	**Assignment 6**
Due: 5/16/11	

#12.4 **Solution.** Let X, Y, Z be the possible dishes that each of the three diners can choose respectively. Since the menu consists of five items we have $|X| = |Y| = |Z| = 5$. The set $X \times Y \times Z$ represents the set of choices possible if we record who selected which dish. Hence, by the multiplication principle (Theorem 10.2.3), there are $|X \times Y \times Z| = |X||Y||Z| = 5^3$ choices.

If we do not care who orders which dish, then there are

(i) 5 ways for the diners to choose one dish from the menu,

(ii) $\binom{5}{2} \cdot 2 = 20$ ways for the diners to choose two dishes from the menu, since one dish is chosen once and the other dish is chosen twice,

(iii) $\binom{5}{3} = 10$ ways for the diners to choose three dishes from the menu.

Hence, in total, there are 35 choices if we do not care who orders which dish. □

#14.1 **Proof.** Since A is finite, we have $A \setminus B$ is too, so write $A \setminus B = \{x_1 \ldots, x_n\}$. Observe that $A \cup B = (A \setminus B) \cup B$ and that $A \setminus B$ and B are disjoint. As B is denumerable, there is a bijection $g : \mathbb{Z}^+ \to B$. Now define a function $h : \mathbb{Z}^+ \to A \cup B$ by

$$h(i) = \begin{cases} x_i & \text{if } \ 1 \leq i \leq n \\ g(i-n) & \text{if } \quad n < i. \end{cases}$$

Clearly h is well-defined, and we now show that h is a bijection. First, we will show that h is an injection. Suppose that $h(i) = h(j)$. Since $h(i) \in A \setminus B$ for $1 \leq i \leq n$ and $h(i) \in B$ for $n < i$, we must have either $1 \leq i, j \leq n$ or $n < i, j$ because $A \setminus B$ and B are disjoint. In the first case, we see $h(i) = h(j)$ implies $i = j$ since $x_1, \ldots, x_n$ are distinct. In the second case, we see $h(i) = h(j)$ implies that $g(i - n) = h(i) = h(j) = g(j - n)$ so $i - n = j - n$ as g is an injection; this implies $i = j$ as desired.

Now we will show that h is a surjection. If $w \in A \cup B$, then $w \in A \setminus B$ or $w \in B$. If $w \in A \setminus B$, then $w = x_i$ for some $1 \leq i \leq n$ so $h(i) = w$. If $w \in B$, then there is $i \in \mathbb{Z}^+$ such that $g(i) = w$ since g is a surjection. Hence $h(i + n) = g(i) = w$. This shows that h is a surjection. We have proved that $h : \mathbb{Z}^+ \to A \cup B$ is a bijection and so $A \cup B$ is denumerable. □

#14.4 **Proof.** Suppose that X and Y are two sets such that $|X| \leq |Y|$ and $|Y| \leq |X|$. We prove that $|X| = |Y|$ by contradiction. Suppose, for a contradiction, that $|X| \neq |Y|$. Then since $|X| \leq |Y|$, we have $|X| < |Y|$, and since $|Y| \leq |X|$, we have $|Y| < |X|$. By definition, if $|X| < |Y|$, then there exists an injection $X \to Y$. By the Cantor-Schroder-Bernstein theorem, however, if $|Y| < |X|$, there does not exist an injection $X \to Y$. Thus we have obtained a contradiction to our assumption that $|X| \neq |Y|$ is false. Hence, $|X| = |Y|$ as required. $\square$

#15.1 (i) We have $100 = 3(33) + 1$ so $q = 33$ and $r = 1$. $\square$

(ii) We have $3 = 100(0) + 3$ so $q = 0$ and $r = 3$. $\square$

#15.2 **Proof.** First suppose that $5 \mid a$. We will show that $5 \mid a^2$. Let $x \in \mathbb{Z}$ be the integer such that $a = 5x$. We have $a^2 = (5x)^2 = 5(5x^2)$ so $5 \mid a^2$.

Now suppose $5 \mid a^2$. We will show that $5 \mid a$. By the division theorem (Theorem 15.1.1), there are unique integers $q, r \in \mathbb{Z}$ such that $a = 5q + r$ and $0 \leq r < 5$. We then have

$$a^2 = (5q + r)^2 = 25q^2 + 10qr + r^2 = 5(5q^2 + 2qr) + r^2.$$

Since $5 \mid a^2$, it follows that $5 \mid r^2$. As $r \in \{0, 1, 2, 3, 4\}$, we must have that $r = 0$. Hence $5 \mid a$. We have shown that an integer a is divisible by 5 if and only if a^2 is divisible by 5. $\square$