MATH 109 **Assignment 7**

Due: 5/23/11

#17.1 **Solution.** By the division algorithm, we have

$$\begin{aligned}
7684 &= 4148(1) + 3536 \\
4148 &= 3536(1) + 612 \\
3536 &= 612(5) + 476 \\
612 &= 476(1) + 136 \\
476 &= 136(3) + 68 \\
136 &= 68(2) + 0
\end{aligned}$$

Rearranging these equations, we find $m = 27$ and $n = -50$.

$$\begin{aligned}
68 &= 476(1) - 136(3) \\
&= 476(1) - (612(1) - 476(1))(3) \\
&= 612(-3) + 476(4) \\
&= 612(-3) + (3536(1) - 612(5))(4) \\
&= 3536(4) + 612(-23) \\
&= 3536(4) + (4148(1) - 3536(1))(-23) \\
&= 4148(-23) + 3536(27) \\
&= 4148(-23) + (7684(1) - 4148(1))(27) \\
&= 7684(27) + 4148(-50). \quad \square
\end{aligned}$$

#17.2 **Solution** By the division algorithm we have

$$\begin{aligned}
7648 &= 4148(1) + 3500 \\
4148 &= 3500(1) + 648 \\
3500 &= 648(5) + 260 \\
648 &= 260(2) + 128 \\
260 &= 128(2) + 4 \\
128 &= 4(32) + 0.
\end{aligned}$$

Rearranging these equations, we find

$$\begin{aligned} 4 &= 260 - 128(2) \\ &= 260 - (648 - 260(2))(2) \\ &= 648(-2) + 260(5) \\ &= 648(-2) + (3500 - 648(5))(5) \\ &= 3500(5) + 648(-27) \\ &= 3500(5) + (4148 - 3500)(-27) \\ &= 4148(-27) + 3500(32) \\ &= 4148(-27) + (7648 - 4148)(32) \\ &= 7648(32) + 4148(-59). \end{aligned}$$

To finish the problem, we see

$$68 = 17 * 4 = 17(7648(32) + 4148(-59)) = 7642(544) - 4148(1003)$$

so $m = 544$ and $n = 1003$. □

#18.1 **Solution** By #17.1, we have

$$272 = 4 * 68 = 4(7684(27) + 4148(-50)) = 7684(108) + 4148(-200),$$

so $m = 108$ and $n = -200$. □

#IV.14 **Proof.** Let $c = am' + bn'$ where $m', n' \in \mathbb{Z}$. Suppose, for a contradiction, that c does not divide a. By the division algorithm, we can write $a = cq + r$ where $0 < r < c$. Observe that $r > 0$ since we have assumed that c does not divide a. Note that

$$0 < r = a - cq = a - (am' + bn')q = a(1 - m'q) + b(n'q) \in \{am + bn : m, n \in \mathbb{Z} \text{ and } am + bn > 0\}.$$

Since $c > r$, this contradicts the definition of c. Hence, our initial assumption is incorrect so c divides a. A similar argument shows that c divides b so c is a common divisor of a and b.

By the previous paragraph $c \leq gcd(a, b)$. Let $x, y \in \mathbb{Z}$ be defined by $gcd(a, b)x = a$ and $gcd(a, b)y = b$. We then have

$$c = am' + bn' = gcd(a, b)xm' + gcd(a, b)yn' = gcd(a, b)(xm' + yn'),$$

so $gcd(a, b)|c$. Since $gcd(a, b), c > 0$, this implies $gcd(a, b) \leq c$. Hence $gcd(a, b) = c = am' + bn'$, which gives an alterate proof of Theorem 17.1.1. □

#IV.15 (i) By the division algorithm, we have

$$\begin{aligned}
252 &= 165(1) + 87 \\
165 &= 87(1) + 78 \\
87 &= 78(1) + 9 \\
78 &= 9(8) + 6 \\
9 &= 6(1) + 3 \\
6 &= 3(2) + 0.
\end{aligned}$$

We see that gcd(252, 165) = 3. We will now rearrange the equations above to write 3 as an integral linear combination of 252 and 165.

$$\begin{aligned}
3 &= 9 - 6 \\
&= 9 - (78 - 9(8)) \\
&= 78(-1) + 9(9) \\
&= 78(-1) + (87 - 78)(9) \\
&= 87(9) - 78(10) \\
&= 87(9) - (165 - 87)(10) \\
&= 165(-10) + 87(19) \\
&= 165(-10) + (252 - 165)(19) \\
&= 252(19) + 165(-29).
\end{aligned}$$

We have $3 = 252(19) + 165(-29)$ so multiplying both sides of this equation by 5 yields

$$15 = 252(95) + 165(-145). \quad \square$$