Let $G_1, G_2$ be groups. We will prove that $G_1 \oplus G_2$ is also a group.

(1) Prove that, for each $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$,
\[(a_1, a_2)(b_1, b_2) \in G_1 \oplus G_2.\]

(2) Prove that, for each $a_1, b_1, c_1 \in G_1$ and $a_2, b_2, c_2 \in G_2$,
\[\left(\left(\left(a_1, a_2\right)\left(b_1, b_2\right)\right)\left(c_1, c_2\right)\right) = \left(a_1, a_2\right)\left(b_1, b_2\right)\left(c_1, c_2\right).\]

(3) Let $e_1$ be the identity of $G_1$ and $e_2$ be the identity of $G_2$. Prove that $(e_1, e_2)$ is the identity of $G_1 \oplus G_2$.

(4) Let $a_1 \in G_1$ and $a_2 \in G_2$. Prove that $(a_1^{-1}, a_2^{-1})$ is the inverse of $(a_1, a_2)$. 
(1) By definition of the componentwise group operation, \((a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2)\). Since \(G_1\) is a group, \(a_1 b_1 \in G_1\); since \(G_2\) is a group, \(a_2 b_2 \in G_2\). Thus, by definition of \(G_1 \oplus G_2\), \((a_1 b_1, a_2 b_2) \in G_1 \oplus G_2\).

(2) Expanding componentwise:
\[
\left((a_1, a_2)(b_1, b_2)\right)(c_1, c_2) = \left((a_1 b_1) c_1, (a_2 b_2) c_2\right)
\]
\[
= \text{Assoc of } G_1, G_2 \left(a_1 (b_1 c_1), a_2 (b_2 c_2)\right)
\]
\[
= (a_1, a_2) \left(b_1, b_2\right)(c_1, c_2).
\]

(3) Let \(a_1 \in G_1\) and \(a_2 \in G_2\). Then
\[
(a_1, a_2)(e_1, e_2) = (a_1 e_1, a_2 e_2) = (a_1, a_2)
\]
\[
(e_1, e_2)(a_1, a_2) = (e_1 a_1, e_2 a_2) = (a_1, a_2)
\]
as required for \((e_1, e_2)\) to be the identity of \(G_1 \oplus G_2\).

(4) We compute
\[
(a_1, a_2)(a_1^{-1}, a_2^{-1}) = (a_1 a_1^{-1}, a_2 a_2^{-1}) = (e_1, e_2)
\]
\[
(a_1^{-1}, a_2^{-1})(a_1, a_2) = (a_1^{-1} a_1, a_2^{-1} a_2) = (e_1, e_2)
\]
as required for \((a_1, a_2)^{-1} = (a_1^{-1}, a_2^{-1})\).