Math 103A Fall 2005 Exam 1

Problem 1 (25 points)

(1a) (15 pts) Let $G$ be a nonempty set with a binary operation, that is, a rule assigning to each pair of elements $(a, b)$ with $a, b \in G$ a new element $ab \in G$. Define what it means for $G$ with this operation to be a group.
(1b) (10 pts) Let $G$ be the set consisting of the 4 elements

$$G = \{Chicago, Houston, Anaheim, SaintLouis\}.$$ 

Abbreviate these elements as $c, h, a, s$ respectively. Define a binary operation on $G$ using the following table:

<table>
<thead>
<tr>
<th></th>
<th>c</th>
<th>h</th>
<th>a</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>c</td>
<td>h</td>
<td>a</td>
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<td>h</td>
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<td>s</td>
<td>s</td>
<td>a</td>
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<td>c</td>
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Here, the element written in row $A$ and column $B$ of the table is the product $AB$. For example, from the table we see that $ha = s$. The binary operation defined above satisfies $(a*b)*c = a*(b*c)$ for all $a, b, c \in G$—you can assume this fact without proof.

Now prove that $G$ is a group under the binary operation given above.
Problem 2 (25 points)

(2a) (15 pts) Calculate gcd(34, 74), using any method you like.
(2b) (10 pts) Find integers \( x, y \in \mathbb{Z} \) such that \( 74x + 34y = 4 \).
Problem 3 (25 points)

(3a) (5 pts) Let $G$ be a group, and let $a \in G$. Give the definition of the centralizer $C(a)$.

(3b) (15 pts) Prove that $C(a)$ is a subgroup of $G$. 
(3c) (5 pts) Let $G$ be the group $\text{GL}(2, \mathbb{R})$, and consider the element

$$a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in G.$$ 

Calculate $C(a)$. 
Problem 4 (25 points)

In this problem, let $G$ be the group $U(12)$.

(4a) (5 pts) List the elements in $G$.

(4b) (15 pts) Find all of the cyclic subgroups of $G$. 

(4c) (5 pts) Is $G$ itself a cyclic group?