Math 103A Fall 2007 Exam 2

November 19, 2007

NAME:

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Problem 1 (25 points)

(a) (5 pts). Clearly state Lagrange’s theorem.
(b) (20 pts) Let $p$ be a prime number. Let $G$ be a group with $|G| = p^n$ for some $n \geq 1$ (such a group is called a $p$-group.) Prove that $G$ has at least one element of order $p$. (Hint: if you don’t know how to start, consider first the special case where $|G| = 9$.)
Problem 2 (25 points)

(a) (10 pts) Let $G$ and $\overline{G}$ be two groups. Define what it means for a function $\phi : G \rightarrow \overline{G}$ to be an isomorphism of groups.
(b) (15 pts) Define the following set of matrices:

\[ G = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{Z} \right\}. \]

The set \( G \) is a group under matrix multiplication (you can assume this.)

Prove that \( G \cong \mathbb{Z} \), in other words that \( G \) is isomorphic to the group of integers with the operation of addition. Hint: you need to find a function \( \phi \) which gives the isomorphism—try something simple.
Problem 3 (20 points)

In this problem, we consider the group $S_7$ of permutations of $\{1, 2, 3, \ldots, 7\}$.

(a) (10 pts). Write the permutation $\alpha = (156)(3547)$ in disjoint cycle form. What is the order of this permutation in the group $S_7$?

(b) (10 pts). Explain why the permutation $\alpha$ in part (a) is an odd permutation. Then find a permutation $\beta \in S_7$ which is an even permutation but which has the same order in $S_7$ as the element $\alpha$. Again briefly explain your answer.
Problem 4 (30 points)

In this problem, consider the following four groups: $A_4$, $\mathbb{Z}_{12}$, $U(21)$, $D_6$. These groups all have order 12 (you don’t have to prove this.) (Note that $D_6$ is the group of all symmetries of a regular hexagon so it contains six rotations and six reflections.)

(a) (10 pts) For each of the four groups, decide if it is Abelian or non-Abelian and list your answers below. Prove your answer only for the alternating group $A_4$. 
(b) (20 pts) Prove that no two of the four groups $A_4$, $\mathbb{Z}_{12}$, $U(21)$, $D_6$ are isomorphic. You can assume without proof all of the basic properties of isomorphisms. (Starting hint: look for some elements of order 2 in $U(21)$.)
(more space for problem 4)