A graph $G = (V, E)$ consists of a vertex set $V$ and an edge set $E$. A graph is used to describe relationships between data points.
A weighted graph $G = (V, E, w)$ is a graph in which each edge $uv$ is associated with some weight $w_{uv} > 0$, $w_{uv} = w_{vu}$. Weighted graphs describe how strong relationships between data points are.
A connection graph $G = (V, E, O, w)$ additionally has a set of rotations: associated with each edge $uv$ is a $d$-dimensional rotation $O_{uv} \in SO(d)$ such that $O_{uv} = O_{vu}^{-1}$.
Applications of connection graphs deal with high-dimensional data sets where scalar weights alone are insufficient to quantify affinities between data points.
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Applications

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- In early work of Chung and Sternberg (1992) [4] connection graphs were used in graph gauge theory for computing the vibrational spectra of molecules examining spins associated with vibrations.
Feature detection

Detect features using SIFT [Lowe, IJCV 2004]
Feature matching

Match features between each pair of images
Applications

The Buckyball Molecule
The Combinatorial Laplacian

For a graph $G$ define the Laplacian $L = D - A$ where $D(u, u) = d_u$ and $A$ is the adjacency matrix.
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We think of $L$ as an operator $\mathcal{F}(G, \mathbb{R}) = \{f : V(G) \to \mathbb{R}\}$, the space of real-valued functions on the vertex set, given by

$$Lf(u) = \sum_{v \in V, u \sim v} w_{uv} (f(u) - f(v)).$$
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$$f^T L f = \sum_{(u,v) \in E} w_{uv}(f(u) - f(v))^2.$$
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Define an operator $\mathbb{L}$ on $\mathcal{F}(G, \mathbb{R}^d)$ by $\mathbb{L} = \mathbb{D} - \mathbb{A}$ where

\[
\mathbb{D}(u, u) = d_u I_{d \times d} \quad \text{and} \quad \mathbb{A}(u, v) = \begin{cases} 
    w_{uv} O_{uv} & (u, v) \in E \\
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We can think of $\mathbb{L}$ as an $n \times n$ matrix whose entries are $d \times d$ matrices, or as an $nd \times nd$ matrix.
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f^T \mathbb{L} f = \sum_{(u, v) \in E} w_{uv} \| O_{uv} f(u) - f(v) \|^2_2.
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A connection graph is consistent if for any cycle \((v_1, v_2, \ldots, v_k, v_1)\) of the underlying graph, 
\[ O_{v_kv_1} \prod_{i=1}^{k-1} O_{v_iv_{i+1}} = I_{d \times d}. \]
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In other words, the product of rotations around any cycle is the identity.

Equivalently, for any two vertices \(u, v\), the product of the rotations along any two distinct paths between \(u\) and \(v\) is the same.
Consistency

Theorem

Let $G$ be a connection graph with dimension $d$ and connection Laplacian $L$. The following are equivalent:

1. $G$ is consistent.
2. For any two vertices $u, v$ and paths $P, P'$ from $u$ to $v$, \[ \prod_{xy \in P} O_{xy} = \prod_{zw \in P'} O_{zw}. \]
3. $0$ is an eigenvalue of $L$ with multiplicity $d$.
4. The eigenvalues of $L$ are the eigenvalues of the Laplacian $L$ of the underlying graph $G$ with multiplicity $d$.
5. For each vertex $v$ we have $O_v \in SO(d)$ such that $O_{uv} = O^{-1}_u O_v$ for all $(u, v) \in E$. 

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Consistency

Theorem

Let $\mathcal{G}$ be a connection graph with dimension $d$ and connection Laplacian $\mathbb{L}$. The following are equivalent:

1. $\mathcal{G}$ is consistent.
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Consistency

Proof:
Construct eigenvectors as follows:

For a fixed vertex $u$, choose a starting vector $x \in \mathbb{R}^d$ and let $f(u) = x$. For $v \sim u$ set $f(v) = O_{uv} f(u)$. Since $G$ is consistent we can continue in this manner defining $f$ on every vertex. Then by construction $f^T L f = \sum_{(u,v) \in E} w_{uv} ||O_{uv} f(u) - f(v)||_2^2 = 0$ so $f$ is an eigenvector with eigenvalue 0.

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Technique 1: For a fixed vertex $u$, choose a frame $O_u$ consisting of $d$ orthonormal vectors.
Then for any vertex $v$, given a path $u = v_0, v_1, \ldots, v_t = v$ from $u$ to $v$, define

$$O_v = O_u \prod_{i=0}^{t-1} O_{v_i v_{i+1}}.$$
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Technique 2: Use the eigenvectors of the Connection Laplacian to create the matrices $O_v$. (Singer, Spielman 2012 [5])
For a graph $G$, denote the transition probability matrix for a random walk on $G$ by $P = D^{-1}A$, and let $Z = \frac{1}{2}(I + P)$ be the matrix for the lazy random walk.
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For a seed probability distribution vector $s$ and jumping constant $0 < \alpha < 1$, a PageRank vector $pr_{\alpha,s} : V \rightarrow \mathbb{R}$ satisfies

$$pr_{\alpha,s} = \alpha s + (1 - \alpha) pr_{\alpha,s} Z.$$
Similarly, define $P = D^{-1}A$ and $Z = \frac{1}{2}(I + P)$ to be the transition probability matrix for the lazy random walk on the connection graph $G$. 
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For an initial seed vector $\hat{s}: V \rightarrow \mathbb{R}^d$, we can define $\hat{pr}_{\alpha,\hat{s}}: V \rightarrow \mathbb{R}^d$ to satisfy the relation

$$\hat{pr}_{\alpha,\hat{s}} = \alpha \hat{s} + (1 - \alpha) \hat{pr}_{\alpha,\hat{s}} Z.$$
A PageRank vector can be used to measure how close a connection graph is to being consistent.
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Define the inconsistency coefficient

\[
\omega_G = \max_{u \in V} \sup_{\hat{\chi}_u} \frac{\sum_{u \in V} ||[\hat{pr}_{\alpha,\hat{\chi}_u} \mathbb{Z}](v)||_2}{\sum_{u \in V} ||\hat{pr}_{\alpha,\hat{\chi}_u}(v)||_2}
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Theorem

\[\omega_G \leq 1\]

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- *If* \( G \) *is consistent, then* \( \omega_G = 1 \).
In a graph, the effective resistance of an edge $e = (u, v)$ is given by

$$R_{eff}(u, v) = 1_e^T B L^+ B^T 1_e$$

where $L^+$ denotes the pseudo-inverse of the Laplacian of the graph, and $B$ is the incidence matrix.
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where $L^+$ denotes the pseudo-inverse of the Laplacian of the graph, and $B$ is the incidence matrix.

Generalize this to connection graphs by defining $\Pi = BL^+ B^T$ and define the connection resistance as

$$R_{\text{eff}}(u, v) = ||\Pi(e, e)||_2.$$
Connection Resistance is used to rank edges in a connection graph.

Edge Ranking Algorithm: For each edge $e$ of $G$, assign to each edge the value $p'_e = w_e R_{eff}(e)$.

Select edges according to probability $p_e$ proportional to $p'_e$. 

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A. Bandeira, A. Singer and D. Spielman, A Cheeger Inequality for the Graph Connection Laplacian, to appear.