Minimum Rank of Outerplanar Graphs

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The Minimum Rank Problem

Given a graph $G$ on $n$ vertices, define $S(G)$ to be the set of $n \times n$ symmetric matrices whose zero-nonzero pattern is given by the edges of $G$. 
The Minimum Rank Problem

Given a graph $G$ on $n$ vertices, define $S(G)$ to be the set of $n \times n$ symmetric matrices whose zero-nonzero pattern is given by the edges of $G$. 

\[
\begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 2 & 1 \\
0 & 0 & 1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 1 & -1 & 0 \\
1 & 0 & -1 & 2 & 0 \\
-1 & -1 & 0 & 5 \\
0 & 0 & 5 & 0 \\
\end{pmatrix} \in S(G)
\]
Given a graph $G$ on $n$ vertices, define $S(G)$ to be the set of $n \times n$ symmetric matrices whose zero-nonzero pattern is given by the edges of $G$.

\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 2 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 & -1 & 0 \\
1 & 0 & -2 & 0 \\
-1 & -2 & 0 & 5 \\
0 & 0 & 5 & 0
\end{bmatrix} \in S(G)
\]
Define the minimum rank of $G$, denoted $mr(G)$, to be the minimum rank over all matrices in $S(G)$. Since $mr(G) + M(G) = n$, computing the minimum rank and the maximum nullity are equivalent problems. Computing $M(G)$ or $mr(G)$ for a given graph is hard.
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Denote by $M(G)$ the maximum nullity of $G$. 
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Computing $M(G)$ or $mr(G)$ for a given graph is hard.
Let $K_n$ be the complete graph on $n$ vertices.
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\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]
Let $K_n$ be the complete graph on $n$ vertices. Then $mr(K_n) = 1$.

Let $S_n$ be the star on $n$ vertices.
Minimum Rank of Some Graphs

- Let $K_n$ be the complete graph on $n$ vertices. Then $mr(K_n) = 1$.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

- Let $S_n$ be the star on $n$ vertices. Then $mr(S_n) = 2$.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Let $C_n$ be the cycle on $n$ vertices.
Let $C_n$ be the cycle on $n$ vertices. Then $mr(C_n) = n - 2$.

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1
\end{bmatrix}
$$
Let $C_n$ be the cycle on $n$ vertices. Then $mr(C_n) = n - 2$. 

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

Idea

Our main idea will be to compute the minimum rank of a graph by covering it with graphs whose minimum rank is known.
Clique Cover Number

Definition

The **clique cover number** of a graph $G$ is the minimum number of cliques needed to cover all the vertices and edges of $G$. It is denoted $cc(G)$. 
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Fact
*If $A$ and $B$ are two $n \times n$ matrices, then $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.***
Clique Cover Number

**Definition**

The **clique cover number** of a graph $G$ is the minimum number of cliques needed to cover all the vertices and edges of $G$. It is denoted $cc(G)$.

**Fact**

*If $A$ and $B$ are two $n \times n$ matrices, then $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.*

**Theorem**

*For any graph $G$, $mr(G) \leq cc(G)$.*
Example

\[ \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 \\
1 & 2 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
\end{pmatrix} \]
Example

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 \\
1 & 2 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}
\]
Example
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\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]
Example

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
, 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
, 
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
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0 & 1 & 1 & 1 & 1 \\
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\end{bmatrix}
\]
Example

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\begin{bmatrix}
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0 & 0 & 0 & 0 & 0 \\
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\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 \\
1 & 2 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]
Example

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\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
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1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 \\
1 & 2 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1
\end{bmatrix}
\]
There are several graphs for which $mr(G) = cc(G)$. A cycle is an example where equality does not hold. $mr(C_n) = n - 2 < n = cc(C_n)$. A star is another example. The clique cover number is way too big. $mr(S_n) = 2 < n - 1 = cc(S_n)$. 
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A star is another example. The clique cover number is way too big.

$$mr(S_n) = 2 < n - 1 = cc(S_n)$$
Definition

Given a graph $G$, a collection $\mathcal{C}$ of subgraphs of $G$ is said to cover $G$ if every edge and every vertex is in some member of $\mathcal{C}$. 

We define the rank sum of a cover, $rs(\mathcal{C})$, to be the sum of the minimum ranks of the graphs in $\mathcal{C}$.

The same proof that $mr(G) \leq cc(G)$ works to show that:

**Lemma**

For any graph $G$ and any cover $\mathcal{C}$ of $G$, $mr(G) \leq rs(\mathcal{C})$. 

M. Kempton (BYU)  
Minimum Rank of Outerplanar Graphs  
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Generalizing the Clique Cover Number

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Generalizing the Clique Cover Number

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The same proof that $mr(G) \leq cc(G)$ works to show that:

Lemma

For any graph $G$ and any cover $\mathcal{C}$ of $G$,

$$mr(G) \leq rs(\mathcal{C})$$
When can we find a cover for a graph using a few simple kinds of graphs whose minimum rank is known that will achieve the minimum rank?
Definition

A graph $G$ is **outerplanar** if it is planar and every vertex is on the exterior face.
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Given a graph $G$, a **$k$-separation** of $G$ is a pair of subgraphs of $G$, $G_1, G_2$ with $k$ vertices in common and with $G = G_1 \cup G_2$. 
Outerplanar Graphs

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![Graph Diagram](image-url)
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**Definition**

Given a graph $G$, a **$k$-separation** of $G$ is a pair of subgraphs of $G$, $G_1, G_2$ with $k$ vertices in common and with $G = G_1 \cup G_2$. 
Theorem (Hsieh; Barioli, Fallat, Hogben)

Let \((G_1, G_2)\) be a 1-separation of \(G\).

\[\text{mr}(G) = \min\{\text{mr}(G_1) + \text{mr}(G_2), \text{mr}(G_1 - v) + \text{mr}(G_2 - v) + 2\}\]
Some Tools

Theorem (Hsieh; Barioli, Fallat, Hogben)

Let \((G_1, G_2)\) be a 1-separation of \(G\).

\[
\text{mr}(G) = \min\{\text{mr}(G_1) + \text{mr}(G_2), \text{mr}(G_1 - v) + \text{mr}(G_2 - v) + 2\}
\]

Theorem (van der Holst)

Let \((G_1, G_2)\) be a 2-separation of \(G\).

\[
\text{mr}(G) = \min\{\text{mr}(G_1) + \text{mr}(G_2),
\quad \text{mr}(G_1 - r_1) + \text{mr}(G_2 - r_1) + 2,
\quad \text{mr}(G_1 - r_2) + \text{mr}(G_2 - r_2) + 2,
\quad \text{mr}(G_1 - R) + \text{mr}(G_2 - R) + 4,
\quad \text{mr}(H_1) + \text{mr}(H_2),
\quad \text{mr}(\overline{G_1}) + \text{mr}(\overline{G_2}) + 2\}
\]
Main Result

**Theorem**

*If $G$ is an outerplanar graph, then there is a cover of $G$ consisting of cliques, stars, and cycles whose rank sum equals the minimum rank of $G$.***
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If $G$ is an outerplanar graph, then there is a cover of $G$ consisting of cliques, stars, and cycles whose rank sum equals the minimum rank of $G$.

Idea of proof: We use induction and the fact that an outerplanar graph always has a 1- or 2-separation, and then apply the formulas.
Theorem

If $G$ is an outerplanar graph, then there is a cover of $G$ consisting of cliques, stars, and cycles whose rank sum equals the minimum rank of $G$.

- Idea of proof: We use induction and the fact that an outerplanar graph always has a 1- or 2-separation, and then apply the formulas.
- This gives a solution to the minimum rank problem for outerplanar graphs.
So far we have been implicitly assuming that the entries of the matrices involved are real numbers.
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We use \( mr^F(G) \) to denote the minimum rank of a graph where the entries of the matrices are taken from the field \( F \).
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We use $mr^F(G)$ to denote the minimum rank of a graph where the entries of the matrices are taken from the field $F$.

Sometimes the field matters in questions of minimum rank, but in many situations, the minimum rank is the same no matter what field we work over. We call such graphs **field independent**.
The formulas for 1-separations and 2-separations work over any field.
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With some tweaking, we can get the proof of our main result to work over any field.
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**Corollary**

*The minimum rank of an outerplanar graph is field independent.*
Open Questions

Question

Are there other classes of graphs whose minimum rank can be given by covers using cliques, stars, and cycles?
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What other kinds of graphs need to be considered in covers to extend this result to larger classes of graphs?
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Question
Do all rank minimizing matrices of an outerplanar graph necessarily come from sums of rank minimizing matrices for graphs in the cover?