REVIEW NOTES FOR MATH 266

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§1.1: Some Basic Mathematical Models; Direction Fields

1. You should be able to match direction fields to differential equations. (see, for example, Problems 15-20).
2. You should be able to recognize the behavior of solutions from a computer generated direction field (e.g., "terminal velocity," see, for example, Problem 25(a,b)).

Recommended: 22, 25(a,b)

§1.2: Solutions of Some Differential Equations

1. Be able to solve $\frac{dy}{dt} = ay + b$, either as a separable equation (see equations (14)-(18)), or using integrating factors.
2. Do not forget the constant of integration when computing an indefinite integral.

Recommended: 17, 19(a-c)

§1.3: Classification of Differential Equations

1. Terminology: ordinary vs. partial differential equations; order; linear vs. nonlinear differential equations.
2. Be able to verify that a given function y is a solution without actually solving the equation.

Recommended: 14, 17

§2.1: Linear Equations; Method of Integrating Factors

1. Be able to recognize a first-order linear differential equation, which can be written as,
$$\frac{dy}{dt} + p(t)y = g(t),$$
and find an integrating factor, $\mu(t) = \exp \int p(t)dt$.
2. Then, multiply the linear equation by the integrating factor,
$$\mu(t)\frac{dy}{dt} + \mu(t)p(t) = \mu(t)g(t),$$
which can be rewritten as,
$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)g(t),$$
integrating both sides yields,
$$\begin{aligned} \mu(t)y &= \int \mu(s)g(s)ds + C \\ y &= \frac{1}{\mu(t)}\left[\int \mu(s)g(s)ds + C\right]. \end{aligned}$$
3. Suggestion: Do not memorize equation 2.1.33; instead, solve enough linear equations to recognize the pattern.
4. Where the solution is defined depends on both the differential equation, and the initial conditions. (see Example 3).

Recommended: 31, 32, **37**.

§2.2: Separable Equations

1. Be able to recognize and separate a separable equation,
$$M(x) + N(y)\frac{dy}{dx} = 0$$
$$\int N(y)dy = \int -M(x)dx + C$$
which is to say that the solution is given by,
$$\int M(x)dx + \int N(y)dy = C.$$
2. *Usually*, one cannot solve a separable equation explicitly for y.
3. Be able to recognize a homogeneous equation, which is of the form $\frac{dy}{dx} = f(x,y)$, where $f(x,y)$ can be expressed as a function of the ratio y/x only. Using the change of variables $v = y/x$, yields $f(x,y) = g(v)$, and the equation becomes,
$$g(v) = f(x,y) = \frac{dy}{dx} = \frac{d}{dx}(vx) = v + x\frac{dv}{dx}.$$
So, we solve $v + x\frac{dv}{dx} = g(v)$, which is a linear differential equation for v in terms of x, and then substitute $v = y/x$ to obtain an expression for y in terms of x.
4. Moral: Change of variables can make equations easier to solve (see supplementary problems B, C).

Recommended: 21, 23

§2.3: Modeling with First Order Equations

1. Know the general procedure for modeling using differential equations:
   (a) Name all quantities, identify independent and dependent variables and constants. Consolidate constants when possible (e.g. in Problem 27, it makes sense to write $C = 6\pi\mu a$).
   (b) Turn the statement (in words) of the problem into a differential equation. Any rate of change should be written as a derivative. Recognize that $a = dv/dt$, $v = dx/dt$.
   (c) Make the differential equation 'friendlier', if possible, by approximating coefficients or ignoring negligible effects.
   (d) Solve the 'friendlier' equation.
   (e) Interpret the solution as a real-world statement.
   (f) Check that the real-world statement makes sense (e.g. check the limit as $t \to \infty$).
2. Solve Examples 2, 3, 4, and become comfortable with similar types of problems.

Recommended: 4, **32**

§2.4: Differences between Linear and Nonlinear Equations

1. Verify that an equation has a unique solution (see Theorem 2.4.1 and Example 1 for linear equations and Theorem 2.4.2 and Example 2 for nonlinear equations).
2. Existence and uniqueness result for a linear equation (Theorem 2.4.1): The solution is defined *at least* on some interval I containing the initial time t_0, where the coefficient functions $p(t)$ and $g(t)$ are continuous (maybe more).
3. Existence and uniqueness result for a nonlinear equation (Theorem 2.4.2): The solution is defined in *some small neighborhood* of the initial conditions if there is a rectangular region about the initial conditions (t_0, y_0) where f and $\partial f/\partial y$ are continuous.

Recommended: 26, 27, 33

§2.5: Autonomous Equations and Population Dynamics

1. Logistic equation $\frac{dy}{dt} = r(1 - y/K)y$.
2. Be able to recognize an autonomous equation $dy/dt = f(y)$.
3. Find equilibrium solutions $y = c$ by looking for roots of $f(y) = 0$.
4. Draw the phase line, and use it to identify where solutions are increasing or decreasing (look at the sign of $f(y)$); and the stability of equilibrium solutions.
5. Compute $\frac{d^2y}{dt^2} = \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt}f = \frac{df}{dy}\frac{dy}{dt} = \frac{df}{dy}f$ to determine points of inflection (when $\frac{d^2y}{dt^2} = 0$) and the concavity of the solutions (check the sign of $\frac{d^2y}{dt^2}$).

6. Draw some solutions freehand.

Recommended: 7, 27

§2.6: Exact Equations and Integrating Factors

1. Given an equation of the form,
$$M(x,y) + N(x,y)\frac{dy}{dx} = 0,$$
check for exactness by seeing if $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is zero.
2. If it is exact, then it satisfies the equations,
$$\frac{\partial \psi}{\partial x} = M(x,y),$$
$$\frac{\partial \psi}{\partial y} = N(x,y),$$
Integrating the first equation yields,
$$\psi = \underbrace{\int M(\xi, y) d\xi}_{Q(x,y)} + h(y).$$
Essentially, in this integral, we treat any y's appearing in M as if they were constants. Substituting this into the second equation yields,
$$N(x,y) = \frac{\partial \psi}{\partial y} = \frac{\partial Q}{\partial y} + \frac{\partial h}{\partial y}$$
$$\frac{\partial h}{\partial y} = N(x,y) - \frac{\partial Q}{\partial y}.$$
We showed that the right hand side of the equation is only a function of y, so we can solve the equation for $h(y)$, and substitute it to obtain $\psi(x,y) = Q(x,y) + h(y)$. Then, $\psi(x,y) = c$ is a solution of the exact differential equation.
3. If the equation is not exact, check to see if it can be made exact by multiplying with an integrating factor of the form $\mu(x)$, $\mu(y)$ or some other specified form (e.g. $\mu(xy)$).
4. *Usually,* one cannot solve an exact equation explicitly for y.

Recommended: 15, 18, 27

§2.7: Numerical Approximations: Euler's Method

1. Euler's method is given by,
$$y_{n+1} = y_n + hf(t_n, y_n).$$
2. Do one or two steps of Euler's method by hand.

Recommended: 15(a) ($t = 1.2$), **21**

§3.1-2: Homogeneous Equations with Constant Coefficients; Fundamental Solutions of Linear Homogeneous Equations

1. Recognize a constant-coefficient homogeneous equation (CCHE).
2. CCHE: $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$ gives the characteristic polynomial $ax^2 + bx + c$.
(The text refers to $ax^2 + bx + c = 0$ as the *characteristic equation.*)
3. If $b^2 - 4ac > 0$, the characteristic equation has real, distinct roots, denoted r_1, r_2, and furthermore, $\{y_1 = e^{r_1 t}, y_2 = e^{r_2 t}\}$ is a fundamental set of solutions for the CCHE.
4. Use an initial condition $y(t_0) = y_0$ and $y'(t_0) = y_0'$ to find the coefficients in the general solution $y = c_1 y_1 + c_2 y_2$.
5. Suggestion: Do not memorize equation 3.2.10. Instead, solve the appropriate system of linear equations on a case-by-case basis.
6. Describe the asymptotic behavior (as $\to \infty$) of solutions.
7. Use the Wronskian $W(y_0, y_1) = y_0 y_1' - y_1 y_0'$ to test whether a pair of solutions form a fundamental set of solutions.

8. Existence and uniqueness for a linear equation (Theorem 3.2.1): The solution is defined *at least* on some interval I (maybe more). This is very similar to the result of Theorem 2.4.1 for first-order linear equations, except that you have to check the continuity of an additional coefficient function.

Recommended: 3.2.13, **3.2.16**

§3.4: Complex Roots of the Characteristic Equation

1. If $z = x + iy$, then $e^z = e^x(\cos y + i \sin y)$ (You may use this and the corresponding facts, like $\frac{d}{dt}(e^{zt}) = ze^{zt}$, without proof.)
2. If $b^2 - 4ac < 0$, then the characteristic equation has complex conjugate roots $\lambda \pm i\mu$, where $\lambda = -\frac{b}{2a}$ and $\mu = \frac{\sqrt{4ac-b^2}}{2a}$. Then, $\{e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)\}$ is a fundamental set of solutions for the CCHE.
3. Describe asymptotic behavior (as $t \to \infty$) of solutions.

Recommended: 19, 25(a-c)

§3.5: Repeated Roots; Reduction of Order

1. Material similar to that in this section appears in §4.2 for the higher-order case.
2. A repeated root of a polynomial is a root with multiplicity greater than 1. Be able to identify when a root of the characteristic polynomial $P(x)$ is repeated:
   (a) By factoring the polynomial $P(x) = a(x - r)^2$.
   (b) By plugging into the characteristic polynomial $P(x)$ and its derivative, $P(r) = 0$ and $P'(r) = 0$.

   In the order two case, only a real root can be repeated.
3. Understand the method of reduction of order for finding a new solution to a homogeneous linear equation when one solution is already known.
4. Be able to use the method of reduction of order to find a new solution to the homogeneous equation $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$ using the solution $y = e^{rt}$ (where r is a repeated root of the characteristic polynomial).
5. If $b^2 - 4ac = 0$, the characteristic equation has real repeated roots r, and $\{e^{rt}, te^{rt}\}$ is a fundamental set of solutions of the CCHE.

Recommended: 17(c,d), 27

§3.6, §4.3: Nonhomogeneous Equations; Method of Undetermined Coefficients

1. If $L[y] = g(t)$ is a nonhomogeneous equation, then the *associated homogeneous equation* is $L[y] = 0$.
2. If Y is a solution of a non-homogenenous equation of order n, and $y_1, \ldots, y_n$ form a fundamental set of solutions for the associated homogeneous equation, then the general solution of the nonhomogeneous equation is $y = Y + c_1y_1 + \cdots c_ny_n$.
3. If $L[y] = a_ny^{(n)} + \ldots + a_1y' + a_0y$ is a constant-coefficient linear differential operator, and $g(t)$ involves only polynomials, exponentials, and sines and cosines, we can make a guess for a particular solution Y:

$g(t)$	$Y(t)$
$P_n(t) = a_nt^n + a_{n-1}t^{n-1} + \ldots + a_0$	$t^s(A_nt^n + A_{n-1}t^{n-1} + \cdots + A_0)$
$P_n(t)e^{\alpha t}$	$t^s(A_nt^n + A_{n-1}t^{n-1} + \cdots + A_0)e^{\alpha t}$
$P_n(t)\begin{cases}\cos\beta t\\ \sin\beta t\end{cases}$	$t^s[(A_nt^n + A_{n-1}t^{n-1} + \cdots + A_0)\cos\beta t$ $+(B_nt^n + B_{n-1}t^{n-1} + \cdots + B_0)\sin\beta t]$
$P_n(t)e^{\alpha t}\begin{cases}\cos\beta t\\ \sin\beta t\end{cases}$	$t^s[(A_nt^n + A_{n-1}t^{n-1} + \cdots + A_0)e^{\alpha t}\cos\beta t$ $+(B_nt^n + B_{n-1}t^{n-1} + \cdots + B_0)e^{\alpha t}\sin\beta t]$

where s is the smallest nonnegative integer that ensures that no term in $Y(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the four cases, s is the number of times (i) 0 is a root of the characteristic equation, (ii) α is a root of the characteristic equation, (iii) $i\beta$ is a root of the characteristic equation, and (iv) $\alpha + i\beta$ is a root of the characteristic equation, respectively.

4. Note that multiplying by too *many* powers of t is as bad as multiplying by too *few*. Thus, we must make sure that we have multiplied by exactly the right power of t.
5. To find the coefficients appearing in the guess for $Y(t)$, plug into the differential equation, and group like terms. This will give a system of linear equations in the unknown coefficients, which can be solved as long as we have multiplied by the appropriate number of powers of t (This corresponds to the s in the table above.)
6. If the problem says 'Find the *form* of a particular solution, but do not determine the constants,' then you do not need to find the constants, but make sure that you multiply by the correct number of powers of t.

Recommended: 3.6.17, 3.6.23(a), 3.6.**29**; 4.3.11, 4.3.18

§3.7: Variation of Parameters

1. Variation of parameters can be used to find a particular solution of a nonhomogeneous equation $L[y] = g(t)$ when $g(t)$ is not of the special form appropriate for the method of undetermined coefficients. To apply the variation of parameters, we need a fundamental set of solutions $\{y_1, y_2\}$ to the associated homogeneous equation.
2. Since it is time-consuming to derive, it is recommended that students memorize one of the following methods:
   (a)
$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$
$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$
   where we solve the first equation for u_1', u_2' integrate to get u_1, u_2, and substitute into the second equation to get a particular solution.
   (b) The particular solution is given by,
$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(t)}dt + y_2(t)\int \frac{y_1(t)g(t)}{W(t)}dt$$
   where $W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the Wronskian of y_1 and y_2, *in that order*. In general $W(y_1, y_2) = -W(y_2, y_1)$.
   Be careful of the following:
   (i) Make sure to take the Wronskian in the proper order.
   (ii) Make sure to put the $(-)$ sign on the correct integral.
   (iii) Make sure that the function in front of the integral ($y_1(t)$, say) is different from the function inside the integral ($y_2(t)$, say).

Recommended: 10, 11, 17

§3.8: Mechanical and Electrical Vibrations

1. Be able to write down the equation of motion for a spring from a word problem. In particular, be able to figure out the values of m, γ and k, and the initial data $u(t_0)$ and $u'(t_0)$ from a word problem.
2. Be able to compute the natural frequency of a spring, and the quasi-frequency and quasi-period of a damped vibrating spring.
3. Be able to rewrite $A\sin(\omega t) + B\cos(\omega t)$ as $R\cos(\omega t - \delta)$.
4. Be able to identify if a spring is critically damped or overdamped.

Recommended: 9, 20

§3.9: Forced Vibrations

1. Be able to identify the stead-state solution and the transient wave for a forced damped vibration.
2. It is recommended that students not attempt to memorize equations (3.9.4, 5) and (3.9.10-13). It is usually easier to solve whatever differential equation arises directly.
3. Be able to describe the behavior of the steady-state response when the frequency ω of the applied force is very large or very small.

4. Understand qualitatively, the phenomenon of resonance (with small damping, the steady-state response to a small external force of the appropriate frequency can have very large amplitutde). *Students need not know formulae (3.9.6, 7).*
5. Be able to solve for the position of a spring experiencing forced undamped vibration. Especially be able to tell whether a spring will experience resonance.

Recommended: 7, 10

§4.1: General Theory of nth Order Linear Equations

1. This is similar to the material in §3.1 and §3.2 for the order-two case.
2. Know the existence and uniqueness result for solutions of higher-order constant-coeffficient homogeneous equations (Theorem 4.1.1).
3. Be able to compute determinants of $n \times n$ matrices.
4. Be able to compute the Wronskian of n solutions of an nth order linear differential equation, and use it to determine whether or not the n solutions form a fundamental set of solutions.
5. If Y is a particular solution of the non-homogenenous equation $L[y] = g(t)$ and $y_1, \dots, y_n$ form a fundamental set of solutions for the associated homogeneous equation $L[y] = 0$, then the general solution of the non-homogeneous equation is $Y + c_1y_1 + \cdots + c_ny_n$.

Recommended: 16, 19(c)

§4.2: Homogeneous Equations with Constant Coefficients

1. This is similar to the material in §3.2, §3.4 and §3.5 for the order-two case.
2. Know the definition of the multiplicity of a root of a polynomial. Be able to compute the multiplicity of a root:
   (a) by factoring the polynomial; or
   (b) by evaluating the polynomial and its derivatives at the root.
3. Be able to perform long division of polynomials.
4. Be able to find a fundamental set of solutions of a constant-coefficient homogeneous equation once the roots (with multiplicities) of its characteristic polynomial are known.
5. Be able to factor many polynomials:
   (a) If $f(x) = a_nx^n + \cdots + a_1x + a_0$ and all the numbers $a_n, \dots, a_1, a_0$ are integers, then every rational root of $f(x)$ is of the form $\pm p/q$ where q divides a_n and p divides a_0.
   (b) If r is a root of $f(x)$ with multiplicity m, then we can consider the roots of $f(x)/(x-r)^m$.
   (c) If $f(x) = x^n - r$, then the roots are $x = (\sqrt[n]{r})e^{i(2\pi k/n)}$ (if $r > 0$) or $x = (\sqrt[n]{s})e^{i(2k+1)\pi/n}$ (if $r = -s < 0$), where $k = 0, 1, \dots, n-1$.

Recommended: 24, 39(b-d)

§6.1: Definition of the Laplace Transform

1. Know the definition of the Laplace transform as the integral

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

   and be able to compute this integral for specific functions f.
2. The inverse Laplace transform of F is the function f so that $\mathcal{L}\{f(t)\} = F(s)$. We compute inverse Laplace transforms using Table 6.2.1.
3. *Need not* be able to verify that the Laplace transform of a function exists (e.g., Theorem 6.1.2). On the final exam, the Laplace transforms will exist (for s sufficiently large) whenever necessary.
4. The Laplace transform is linear, so $\mathcal{L}\{c_1f_1(t) + c_2f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$.

Recommended: 10,15

§6.2: Solution of Initial Value Problems

1. A table of Laplace transforms will be available during the exam, but it is recommended that you become as familiar as possible with Table 6.2.1. This will allow you to work more efficiently and should cut down on mistakes. (You can always use the table to check your solutions.)
2. Table 6.2.1 gives $\mathcal{L}\{f(t)\}$ for many functions f. Be able to use this chart to compute $\mathcal{L}^{-1}\{F(s)\}$, the inverse Laplace transform of F for many functions F. This is, given a function F, be able to recognize it in the right-hand column, so that $f(t) = \mathcal{L}^{-1}\{F(s)\}$ can be read off the left-hand column.

3. Know the relation $\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$. This can be used to compute Laplace transforms of higher derivatives as well. For example,
$$\mathcal{L}\{y''\} = s\mathcal{L}\{y'\} - y'(0) = s^2\mathcal{L}\{y\} - sy(0) - y'(0).$$
4. Upon taking the Laplace transform, the constant-coefficient linear differential equation
$$a_n\frac{d^ny}{dt^n} + \cdots + a_1\frac{dy}{dt} + a_0y = g(t)$$
becomes the algebraic equation
$$(a_ns^n + \cdots + a_1s + a_0)\mathcal{L}\{y\} - \text{junk} = G(s)$$
where 'junk' depends on the initial conditions and $G(s) = \mathcal{L}^{-1}\{g(t)\}$. The expression in front of $\mathcal{L}\{y\}$ is the characteristic polynomial.
5. Be able to solve for y in
$$\mathcal{L}\{y\} = \frac{\text{junk}}{\text{char. poly.}} + \frac{G(s)}{\text{char. poly.}}$$
by computing the inverse Laplace transform of the right-hand side. Some tricks:
   (a) If the denominator on the right-hand side can be factored, use partial fractions to break up the right-hand side into expressions with lower degree denominators.
   (b) If the denominator on the right-hand side cannot be factored, complete the square in the denominator. This usually produces an expression in which the variable s has been shifted; then see below (d). (This works if the denominator can be factored, too.)
   (c) If an expression on the right-hand side involves an exponential term, first compute the inverse Laplace transform without the exponential, then shift the answer (See §6.3.)
   (d) If an expression on the right-hand side is familiar except that the variable s has been shifted, first compute the inverse Laplace transform without the shift, then multiply the answer by an exponential term (See §6.3.)

Recommended: 8, 17

§6.3: Step Functions

1. Know the notation
$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$
This is the *unit step function* (at $t = c$).
2. For any function f, the graph of $g(t) = u_c(t)f(t-c)$ is the same as that of f, but shifted c units to the right.
3. Know the behavior of functions and their Laplace transforms under shifting. Suppose that $F(s) = \mathcal{L}\{f(t)\}$.
   (a) $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$. That is, shifting the function in the time domain, multiplies the Laplace transform by an exponential. This is equivalent to the statement that $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$.
   (b) $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$. That is, multiplying the function by an exponential shifts the Laplace transform. Notice that the sign of the exponent here is different. This is equivalent to the statement that $\mathcal{L}^{-1}\{F(s-c)\} = e^{ct}f(t)$.
   (c) Be able to describe a piecewise-defined function as a combination of step functions, and *vice versa*.
   (d) Given an expression $u_c(t)g(t)$, be able to rewrite $g(t) = h(t-c)$. For example, $u_1(t)(t^2 - 2t + 2)$ can be rewritten as $u_1(t)((t-1)^2 + 1) = u_1(t)h(t-1)$ where $h(t) = t^2 + 1$ *. Then we have,
$$\mathcal{L}\{u_1(t)((t-1)^2 + 1)\} = e^{-t}\mathcal{L}\{t^2 + 1\} = e^{-t}\Big(\frac{2}{s^3} + \frac{1}{s}\Big).$$

*In general, to obtain $h(t)$ such that $h(t-c) = g(t)$, we let $u = t - c$, which implies $t = u + c$, and evaluate $g(t)$ at $t = u + c$, which yields an expression in u, which is precisely $h(u)$.

(e) To compute (say) $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+4}\right\}$, first compute the answer without the exponential term in the frequency domain, $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin(2t)$. Then shift this answer by the amount indicated in the exponential,

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+4}\right\} = u_3(t)\Big(\frac{1}{2}\sin(2(t-3))\Big).$$

(f) To compute (say) $\mathcal{L}^{-1}\left\{\frac{1}{(s+5)^2+1}\right\}$, first compute the inverse for the unshifted function $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$, and then multiply the answer by an exponential with the power indicated by the shifted variable $(s+c)$.

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+5)^2+1}\right\} = e^{5t}\sin(t).$$

Recommended: 8, 9, 18.

§6.4: Differential Equations with Discontinuous Forcing Functions

1. Given a discontinuous forcing function of the form

$$g(t) = \begin{cases} g_0(t), & 0 \le t < t_1 \\ g_1(t), & t_1 \le t < t_2 \\ \vdots & \vdots \\ g_n(t), & t_n \le t < \infty \end{cases}$$

we express it in terms of step functions as,

$$g(t) = g_0(t)(1-u_{t_1}(t)) + \left[\sum_{i=1}^{n-1} g_i(t)(u_{t_i}(t) - u_{t_{i+1}}(t))\right] + g_n(t)u_{t_n}(t).$$

Then, by rewriting each of these terms into the form $u_c(t)f(t-c)$, we are able to compute the Laplace transform using the identity $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$.

2. This allows us to convert a constant-coefficient differential equation into an equation of the form

$$\mathcal{L}\{y\} = \frac{\text{junk}}{\text{char. poly.}} + \frac{G(s)}{\text{char. poly.}}$$

as in §6.2 above, even if the right-hand side $g(t)$ is piecewise defined. This section just applies the techniques of §6.3 to solving differential equations.

Recommended: 10, 15.

§6.5: Impulse Functions

1. $\delta(t)$ denotes an impulse function at time $t=0$, and $\delta(t-c)$ denotes an impulse function at $t=c$.
2. Know that $\mathcal{L}\{\delta(t-c)\} = e^{-cs}$.
3. Be able to solve constant-coefficient differential equations when the right-hand side is an impulse function, $g(t) = \delta(t-c)$, by using the formula for the Laplace transform of an impulse function.
4. If the right-hand side has the form $g(t) = h(t)\delta(t-c)$, it can be replaced with $g(t) = h(c)\delta(t-c)$, where the function $h(t)$ has been replaced with the evaluation of the function at the point $t=c$. In particular, this means that $\mathcal{L}\{h(t)\delta(t-c)\} = e^{-cs}h(c)$. This is because $\delta(t-c)$ is zero everywhere except $t=c$.

Recommended: 1, 7, 13.

§6.6: The Convolution Integral

1. Know the definition of a convolution as an integral

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

2. If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, then $F(s)G(s) = \mathcal{L}\{(f*g)(t)\}$.
3. To find the Laplace transform of an integral $\int_0^t$ something $d\tau$, where the integrand 'something' involves both t and τ, find two expressions in u such that if we substitute $u = t-\tau$ in the first expression, and $u=\tau$ in the second expression, and then multiply the two expressions, we get

the integrand. For example, for the integral $\int_0^t (t-\tau+1)^2\cos(2\tau)d\tau$, the two expressions are $f(u)=(u+1)^2$, and $g(u)=cos(2u)$. Then,

$$\int_0^t (t-\tau+1)^2\cos(2\tau)d\tau = (t+1)^2 * \cos(2t)$$

and

$$\mathcal{L}\left\{\int_0^t (t-\tau+1)^2\cos(2\tau)d\tau\right\} = \mathcal{L}\{(t+1)^2\}\mathcal{L}\{\cos(2t)\}$$

4. If a constant-coefficient differential equation transforms, upon taking the Laplace transform into an equation of the form $\mathcal{L}\{y\} = P(s)\cdot Q(s)$, and we know the inverse Laplace transform of $P(s)$ and $Q(s)$, then we can express y as a convolution $\mathcal{L}^{-1}\{P(s)\} * \mathcal{L}^{-1}\{Q(s)\}$. For example, if $\mathcal{L}\{y\} = \frac{2}{s^3}\cdot\frac{1}{s^2+1}$, then $y = t^2 * \sin(t)$.
5. *Need not* know the concept of a transfer function and impulse response.

Recommended: 5, 8.

§7.1-7.3: Matrices, Systems of Linear Algebraic Equations, Linear Independence, Eigenvalues and Eigenvectors

1. The material in these chapters are used as background for the rest of Chapter 7, and will not be explicitly tested in the exam. If you can do the problems in the other sections, you do not need to worry about these sections.

§7.4: Basic Theory of Systems of First Order Linear Equations

1. Be able to test whether solutions $\vec{x}^{(1)},\ldots,\vec{x}^{(n)}$ to a homogeneous system of n equations involving n functions $x_1,\ldots x_n$ form a fundamental system by computing the Wronskian

$$W(\vec{x}^{(1)},\ldots,\vec{x}^{(n)}) = \det\left[\vec{x}^{(1)}|\cdots|\vec{x}^{(n)}\right].$$

If they form a fundamental system of solutions, then the general solution is

$$vecx = c_1\vec{x}^{(1)}+\cdots+c_n\vec{x}^{(n)}.$$

2. Be able to find a particular solution satisfying a given initial condition from the general solution. For example, if $\vec{x}^{(1)} = \begin{bmatrix}\sin t\\ e^t\end{bmatrix}$ and $\vec{x}^{(2)} = \begin{bmatrix}\cos t\\ t^2+1\end{bmatrix}$ and we wish to satisfy the initial condition $\vec{x}(0) = \begin{bmatrix}3\\2\end{bmatrix}$, then we are trying to find c_1 and c_2 such that

$$\begin{bmatrix}3\\2\end{bmatrix} = c_1\begin{bmatrix}\sin(0)\\ e^0\end{bmatrix} + c_2\begin{bmatrix}\cos 0\\ (0)^2+1\end{bmatrix} = \begin{bmatrix}c_2\\ c_1+c_2\end{bmatrix}.$$

Which is a system of equations in c_1 and c_2.

Recommended: 6(a), 7(a).

§7.5: Homogeneous Linear Systems with Constant Coefficients

1. A *constant-coefficient homogeneous system of linear differential equations* can be written in the form $\vec{x} = \mathbf{A}\vec{x}$ where $\mathbf{A}$ contains the coefficients of the system. To solve such a differential equation, we perform the following steps:
   (a) Find the characteristic polynomial of $\mathbf{A}$, given by $\det(\mathbf{A}-r\mathbf{I})$, and is a degree n polynomial in r. If $\mathbf{A} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is a 2×2 matrix, we have,

   $$\det(\mathbf{A}-r\mathbf{I}) = \det\begin{bmatrix}a-r & b\\ c & d-r\end{bmatrix} = (a-r)(d-r)-bc = r^2-(a+d)r+(ad-bc).$$

   Notice that the coefficient of r is $-(a+d)$ and not $a+d$. This recipe does not work for larger matrices.
   (b) Factor the characteristic polynomial of $\mathbf{A}$ to compute its eigenvalues and their algebraic multiplicities.
   (c) For each eigenvalue r, find the corresponding eigenvector(s) by solving the system of linear equations $(\mathbf{A}-r\mathbf{I})\vec{\xi} = \vec{0}$. If the number of eigenvectors one obtains is less than the multiplicity of the eigenvalue, see §7.8.

(d) If $r_1, \ldots, r_n$ are the eigenvalues and $\vec{\xi}_1, \ldots, \vec{\xi}_n$ are the corresponding eigenvectors, then the general solution is

$$\vec{x} = c_1 e^{r_1 t}\vec{\xi}_1 + \cdots + c_n e^{r_n t}\vec{\xi}_n.$$

If any of the eigenvalues are complex, then this solution will also be complex. See §7.6.

2. Although you should be able to deal with systems of equations which are larger than 2×2, most of the problems will be concerned with the 2×2 case. In this case, we have two equations involving the two functions x_1 and x_2. The x_1x_2-plane is called the *phase plane*. A *phase portrait* is a parametric plot of several solutions $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ in the phase plane.
3. Be able to sketch the phase portrait once you know the eigenvalues and eigenvectors. Also be able to deduce information about the eigenvalues and eigenvectors from a phase portrait.
4. Know the descriptions of equilibrium solutions. If the eigenvalues r_1, r_2, are both nonzero, then the only equilibrium solution is $\vec{x} = \vec{0}$.
   (a) If r_1 and r_2 are real, distinct, and of opposite sign, then $\vec{x} = \vec{0}$ is a saddle point. See Figure 7.5.2(a). Then $\vec{x} = \vec{0}$ is asymptotically unstable.
   (b) If r_1 and r_2 are real, distinct, and of the same sign, then $\vec{x} = \vec{0}$ is a node. See Figure 7.5.4(a). Then $\vec{x} = \vec{0}$ is asymptotically unstable (if r_1 and r_2 are positive) or asymptotically stable (if r_1 and r_2 are negative).
   (c) If $r_1 = r_2$ is a repeated root with only one eigenvector, then $\vec{x} = \vec{0}$ is an improper node. See Figure 7.8.2(a). Then $\vec{x} = \vec{0}$ is asymptotically unstable (if r_1 is positive) or asymptotically stable (if r_1 is negative).
   (d) If $r_1 = i\mu$ and $r_2 = -i\mu$, then $\vec{x} = \vec{0}$ is a center. Then, $\vec{x} = \vec{0}$ is stable, but not asymptotically stable.
   (e) If $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, where $\lambda \neq 0$ and $\mu \neq 0$, then $\vec{x} = \vec{0}$ is a spiral point. See Figure 7.6.2(a). Then $\vec{x} = \vec{0}$ is asymptotically unstable (if λ is positive) or asymptotically stable (if λ is negative).

Recommended: 5, 16, 25(a, b).

§7.6: Complex Eigenvalues

1. if $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, then we only need to find an eigenvector $\vec{\xi}_1$ of $\mathbf{A}$ corresponding to eigenvalue r_1. This is because if $\vec{\xi}_1$ is an eigenvector with eigenvalue r_1, then its complex conjugate $\vec{\xi}_2 = \overline{\vec{\xi}_1}$ is an eigenvector with eigenvalue $r_2 = \overline{r}_1$.
2. In this case, the general solution $c_1 e^{r_1 t}\vec{\xi}_1 + c_2 e^{r_2 t}\vec{\xi}_2$ is complex. Write $e^{r_1 t} = e^{\lambda t}(\cos(\mu t) + i\sin(\mu t))$ and multiply out $\vec{\xi}_1 e^{r_1} t$ (using $\vec{\xi}_2$ and r_2 would work as well). The real and imaginary parts of the product form a fundamental system of solutions to the system of linear differential equations.

Recommended: Example 2, 5.

§7.8: Repeated Eigenvalues

1. If $r_1 = r_2$ is a repeated root with only one eigenvector, we still have one eigenvector $\vec{\xi}_1$ which gives a solution of the form $\vec{x}^{(1)} = e^{r_1 t}\vec{\xi}_1$. **The second solution is <u>not</u> just** $te^{r_1 t}\vec{\xi}_1$. Instead, the second solution is

$$\vec{x}^{(2)} = te^{r_1 t}\vec{\xi}_1 + e^{r_1 t}\vec{\eta},$$

where $(\mathbf{A} - r_1\mathbf{I})\vec{\eta} = \vec{\xi}$. This is the same equation we would solve to find an eigenvector, except that the right-hand side is $\vec{\xi}$ and not $\vec{0}$.

Recommended: 1, 7.

§7.9: Nonhomogeneous Linear Systems

1. We will consider constant-coefficient nonhomogenenous systems of linear differential equations. These can be written in the form $\vec{x}' = \mathbf{A}\vec{x} + \vec{g}(t)$.
2. If $\vec{g}(t)$ involves only sines, cosines, exponentials, and non-negative powers of t, then we can use the method of undetermined coefficients. The method is almost identical to that discussed in §3.6 and §4.3, with two subtle differences:
   (a) The 'constants' by which we multiply our guesses are vectors, not scalars.
   (b) If we have to multiply a gues, say e^{3t} by t, then our guess will be $\vec{a}te^{3t} + \vec{b}e^{3t}$, not just $\vec{a}te^{3t}$.

Once we have made our guess, we plug it into the differential equation to get a system of linear equations in the entries of our constant vectors, which we can solve to determine those vectors.

3. Suppose $\mathbf{A}$ is a $n \times n$ matrix. If these is a full set n linearly independent eigenvectors of length n, we can construct an $n \times n$ matrix $\mathbf{T}$ whose columns are these vectors.
   (a) $\mathbf{T}$ is invertible, and $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is a diagonal matrix. We can construct a new system of equations, $\vec{y}' = \mathbf{B}\vec{y} + \vec{h}(t)$, where $\mathbf{B}$ was defined above, and $\vec{h} = \mathbf{T}^{-1}\vec{g}(t)$.
   (b) This new system of equations are n decoupled equations, which can be solved individually. A typical equation in this system of equations is $y_3' = r_3 y_3 + h_3(t)$, where r_3 is the third eigenvalue. This is a linear equation involving only y_3. We can solve it for y_3 after multiplying it by the integrating factor $e^{-r_3 t}$.
   (c) Now recover the solution $\vec{x}$ by putting $\vec{x} = \mathbf{T}\vec{y}$.

Recommended: 1 (using diagonalization), 8 (using the method of undetermined coefficients).

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These notes are a slightly modified and expanded version of Loren Spice's review handouts for a previous semester of Math 266, which are available at `http://www.math.purdue.edu/~lspice/2004MA266.html`.

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