MATH 366 FINAL EXAM SOLUTIONS

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(1) (a) We have that $M = y$, and $N = (2x - ye^y)$. By direct computation we have,

$$M_y - N_x = 1 - 2 = -1 \neq 0,$$

so the equation is not exact.

(b) Multiplying by the integrating factor $\mu = y$, yields $\mu M = y^2$, $\mu N = (2xy - y^2 e^y)$, and by direct computation,

$$(\mu M)_y - (\mu N)_x = 2y - 2y = 0,$$

and hence multiplying the equation by the integrating factor transforms the equation into an exact one.

(c) We have,

$$\frac{\partial \phi}{\partial x} = y^2,$$
$$\frac{\partial \phi}{\partial y} = 2xy - y^2 e^y,$$

from which we obtain,

$$\phi(x, y) = \int y^2 dx + h(y) = xy^2 + h(y).$$

Substitution into the second equation yields,

$$2xy - y^2 e^y = \frac{\partial \phi}{\partial y} = 2xy + \frac{\partial h}{\partial y}$$
$$\frac{\partial h}{\partial y} = -y^2 e^y$$
$$h = \int -y^2 e^y dy = -y^2 e^y + \int 2ye^y dy = -y^2 e^y + 2ye^y - \int 2e^y dy = e^y(-y^2 + 2y - 2)$$

Hence, we have that the general solution is given by,

$$\phi(x, y) = xy^2 + e^y(-y^2 + 2y - 2) = c$$

(2) We have that $W(t, g) = t^2 e^t$, which means,

$$\begin{vmatrix} t & g \\ 1 & g' \end{vmatrix} = tg' - g = t^2 e^t$$

A first-order differential equation which is satisfied by g is,

$$tg' - g = t^2 e^t$$

which can be rewritten as,

$$g' - \frac{1}{t} g = te^t$$

which has an integrating factor, $\mu = e^{\int -\frac{1}{t} dt} = e^{-\ln t} = \frac{1}{t}$. From this, we have,

$$\frac{d}{dt}\left(\frac{g}{t}\right) = e^t$$
$$\frac{g}{t} = e^t + c$$
$$g = te^t + ct$$

(3) The homogeneous equation is $y''+y=0$, with the characteristic equation $r^2+1=0$, which has roots $r=\pm i$. This corresponds to solutions $\sin t$ and $\cos t$. The form of the particular solution should be $Y(t)=t^{s_1}(At+B)+t^{s_2}[(Ct+D)\sin t+(Et+F)\cos t)]$, where $s_1=\#$ of roots at $0=0$, and $s_2=\#$ of roots at $i=1$. Hence, a suitable form of the particular solution is,

$$Y(t)=At+B+(Ct^2+Dt)\sin t+(Et^2+Ft)\cos t$$

(4) (a) $y_1(t)=t$, $y_1'(t)=1$, $y_1''(t)=0$. Hence, $t^2y_1''-t(t+2)y_1'+(t+2)y_1=-t(t+2)+(t+2)t=0$, which verifies that $y_1(t)=t$ is a solution of the homogeneous equation.
$y_2(t)=te^t$, $y_2'=e^t(1+t)$, $y_2''=e^t(2+t)$. Hence, $t^2y_2''-t(t+2)y_2'+(t+2)y_2=t^2(2+t)e^t-t(t+2)(1+t)e^t+(t+2)te^t=(2t^2+t^3-t^3-3t^2-2t+t^2+2t)e^t=0$, which verifies that $y_2(t)=te^t$ is a solution of the homogeneous equation.

(b) We first compute the Wronskian,

$$W(y_1,y_2)=\begin{vmatrix} t & te^t \\ 1 & e^t(1+t)\end{vmatrix}=t^2e^t$$

We have that,

$$\begin{aligned}\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2'\end{bmatrix}\begin{bmatrix} u_1' \\ u_2'\end{bmatrix}&=\begin{bmatrix} 0 \\ 2t\end{bmatrix}\\ \begin{bmatrix} u_1' \\ u_2'\end{bmatrix}&=\frac{1}{W(y_1,y_2)}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1\end{bmatrix}\begin{bmatrix} 0 \\ 2t\end{bmatrix}\\ &=\frac{1}{t^2e^t}\begin{bmatrix} -2t^2e^t \\ 2t^2\end{bmatrix}\\ &=\begin{bmatrix} -2 \\ 2e^{-t}\end{bmatrix}\end{aligned}$$

From this, we have that $u_1=-2t$ and $u_2=-2e^{-t}$, and we obtain a particular solution of the form,

$$Y(t)=u_1y_1+u_2y_2=-2t(t)-2e^{-t}(te^t)=-2t^2-2t$$

The general solution is given by,

$$y(t)=c_1t+c_2te^t-2t^2$$

(5) The roots of the characteristic equation are 0, 1, 2, and $\pm i$, which implies that the characteristic equation is given by,

$$r(r-1)(r-2)(r+i)(r-i)=r^5-3r^4+3r^3-3r^2+2r$$

which corresponds to the following linear differential equation,

$$y^{(5)}-3y^{(4)}+3y^{(3)}-3y^{(2)}+2y'=0$$

(6) We wish to construct a power series solution of the form,

$$y(x)=\sum_{n=0}^{\infty}a_n(x-1)^n.$$

Computing the derivatives, we have,

$$y'(x)=\sum_{n=1}^{\infty}na_n(x-1)^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n$$

Substituting into the differential equation yields,

$$\begin{aligned}
0 &= y'' - xy' - y \\
&= y'' - (x-1)y' - y' - y \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - (x-1)\sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=0}^{\infty} na_n(x-1)^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=0}^{\infty}(n+1)[(n+2)a_{n+2} - a_{n+1} - a_n](x-1)^n
\end{aligned}$$

The recurrence relation is given by,

$$(n+2)a_{n+2} - a_{n+1} - a_n = 0$$

The two linearly independent solutions are given by,

Taking $a_0 = 1, a_1 = 0$,

$$y_1(x) = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots$$

Taking $a_0 = 0, a_1 = 1$,

$$y_2(x) = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots$$

(7) The indicial equation is given by,

$$r(r-1) - 2r + 3 = r^2 - r - 2r + 3 = r^2 - 3r + 3$$

The roots are given by,

$$r = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 3}}{2} = \frac{1}{2}(3 \pm i\sqrt{3})$$

The general solution is given by,

$$y(x) = |x|^{3/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln(x)\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln(x)\right)\right]$$

(8) (a) Rewriting the equation yields,

$$y'' + \frac{1}{x}y = 0$$

which means that $p(x) = 0$ and $q(x) = \frac{1}{x}$. $q(x)$ is singular at $x = 0$, and it remains to show that $x = 0$ is a regular singular point. We compute the following limits,

$$\begin{aligned}
p_0 &= \lim_{x\to 0} xp(x) = 0 \\
q_0 &= \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x = 0
\end{aligned}$$

This establishes that the singular point at $x = 0$ is regular.

(b) From the previous part, we obtained that the exponents at the singularity are $p_0 = 0$, $q_0 = 0$. The corresponding indicial equation is,

$$r(r-1) + p_0 r + q_0 = r(r-1),$$

which has roots $r = 0, 1$.

(c) We look for a solution of the form, $y_1(x) = x\,[1 + \sum_{n=1}^{\infty} a_n x^n]$. Then, $y_1''(x) = \sum_{n=1}^{\infty}(n+1)na_n x^{n-1}$, and direct substitution yields,

$$\begin{aligned}
0 &= xy_1'' + y_1 \\
&= \sum_{n=1}^{\infty}(n+1)na_n x^n + x + \sum_{n=1}^{\infty} a_n x^{n+1} \\
&= \sum_{n=1}^{\infty}(n+1)na_n x^n + x + \sum_{n=2}^{\infty} a_{n-1} x^n \\
&= (2a_1+1)x + \sum_{n=2}^{\infty}[(n+1)na_n + a_{n-1}]x^n
\end{aligned}$$

This gives the recurrence relation,

$$a_n = \frac{-a_{n-1}}{n(n+1)}$$

where $a_0 = 1$. This yields one of the solutions,

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$$