MATH 170B HOMEWORK 1 SOLUTIONS

§3.1: Questions 4, 5, 7

4. By theorem, $|r - c_n| \leq 2^{-(n+1)}(b_0 - a_0) \leq \varepsilon$. So, $-(n+1)\log 2 + \log(b_0 - a_0) \leq \log\varepsilon \Rightarrow -(n+1) \leq [\log\varepsilon - \log(b_0 - a_0)]/\log 2$.
   Hence, $n > [\log(b_0 - a_0) - \log\varepsilon]/(\log 2) - 1$.

5. Relative precision $= |r - c_n|/|r| \leq \varepsilon$. Since $r \geq a_0 > 0$, we require

   $|r - c_n|/a_0 \leq \varepsilon$, $2^{-(n+1)}(b_0 - a_0)/a_0 \leq \varepsilon$, $-(n+1) \leq \log[(\varepsilon a_0)/(b_0 - a_0)]/\log 2$,

   $n \geq \log[(b_0 - a_0)/(\varepsilon a_0)]/(\log 2) - 1 = [\log(b_0 - a_0) - \log\varepsilon - \log a_0]/(\log 2) - 1$

7. By Problem 3.1.4, $|r - c_n| \leq \varepsilon$ after $n \geq [\log(b_0 - a_0) - \log\varepsilon]/(\log 2) - 1$. Here $n \geq 6/(\log 2) - 1 = 18.93$. So in 19 steps, we obtain 10^{-6} absolute accuracy. On `MARC-32`, machine precision 2^{-24} is obtained in $n > (24\log 2)/(\log 2) - 1 = 23$ steps for absolute accuracy to full precision.

   By Problem 3.1.5, relative precision $|r - c_n|/|r| \leq \varepsilon$ requires $n \geq [\log(b_0 - a_0) - \log\varepsilon - \log a_0)]/(\log 2) - 1$. Here $n \geq [6 - \log 2]/(\log 2) - 1 = [6/\log 2] - 2 = 17.93$. So in 18 steps, we obtain 10^{-6} relative accuracy. On the `MARC-32`, we have $n \geq [24\log 2 - \log 2]/(\log 2) - 1 = 24 - 2 = 22$ steps for relative accuracy to full precision.

§3.2: Questions 5, 10, 15, 19*, 23(a)

5. Let $\lim_{n\to\infty} x_n = r$, then $r = 2r - r^2 y \Rightarrow r = 1/y$. So the purpose of the formula is to compute $1/y$. Now $x_{n+1} = 2x_n - x_n^2 y = x_n + (x_n - x_n^2 y) = x_n + x_n^2(1/x_n - y) = x_n + (x_n^{-1} - y)/x_n^{-2}$. So $x_{n+1} = x_n - (x_n^{-1} - y)/(-x_n^{-2})$. Thus, it is Newton's iteration for $f(x) = x^{-1} - y$.

   Alternative Solution: $x_{n+1} = 2x_n - x_n^2 y = x_n - f(x_n)/f'(x_n)$. Change y to a and $f(x)$ to y. Solve $x - ax^2 = -y/y'$. Now $(\log y)' = y'/y = 1/[x(ax-1)] = A/x + B/(ax-1)$ implies $A = -1$ and $B = a$. So $(\log y)' = -1/x + a/(ax-1)$ or $\log y = -\log x + \log(ax-1) = \log[(ax-1)/x]$ implies $y = a - 1/x$. Hence, original problem to solve $f(x) = y - 1/x$.

10. $f(x) = x^3 - R$; $f'(x) = 3x^2$. Thus, the Newton's iteration formula for computing $\sqrt[3]{R}$ is: $x_{n+1} = x_n - (x_n^3 - R)/3x_n^2 = (2x_n + R/x_n^2)/3$. For $x > 0$, $f'(x) > 0$ and $f''(x) > 0$. Then by Theorem 2 the Newton iteration will converge from any point > 0. For $x < 0$, Newton method will not always converge since one of the iterates could not be the origin. A quick calculation shows that if we start with the point $x = \sqrt[3]{-R/2}$, then the first iterate will be the point $x_1 = 0$ where $f'(x_1) = 0$ and $x_2 = \infty$. So the method fails in this case. Hence, it converges for all $x > 0$.

15. $e_{n+1} = e_n - f(x_n)/f'(x_0)$. Also $0 = f(r) = f(x_n - e_n) = f(x_n) - e_n f'(\xi_n) \Rightarrow f(x_n) = e_n f'(\xi_n)$. Hence, $e_{n+1} = e_n(1 - f'(\xi_n)/f'(x_0)) \Rightarrow s = 1$, $C = 1 - f'(\xi_n)/f'(x_0)$.

19. $f(r) = f'(r) = \cdots = f^{(k-1)}(r) = 0 \neq f^{(k)}(r)$.
$f(x_n) = f(r+e_n) = f(r)+e_nf'(r)+\cdots+e_n^{(k-1)}f^{(k-1)}(r)/(k-1)!+e_n^kf^{(k)}(r)/k! +e_n^{k+1}f^{(k+1)}(\xi_n)/(k+1)! \Rightarrow f(x_n) = e_n^kf^{(k)}(r)/k! + e_n^{k+1}f^{(k+1)}(\xi_n)/(k+1)!$.

Similarly, $f'(x_n) = e_n^{k-1}f^k(r)/(k-1)! + e_n^kf^{(k+1)}(\eta_n)/k!$. Then $e_{n+1} = x_{n+1} - r = x_n - r - kf(x_n)/f'(x_n) = e_n - [e_n^kf^{(k)}(r)/(k-1)! +e_n^{k+1}f^{(k+1)}(\xi_n)/(k+1)(k-1)!]/[e_n^{k-1}f^{(k)}(r)/(k-1)! + e_n^kf^{(k+1)}(\eta_n)/k!] = e_n^2[f^{(k+1)}(\eta_n)/k! - f^{(k+1)}(\xi_n)/(k+1)(k-1)!]/[f^{(k)}(r)/(k-1)! + e_nf^{(k+1)}(\eta_n)/k!]$, implying quadratic convergence.

Alternative Solution: $f^{(j)}(r) = 0$ for $0 \le j \le m-1$ and $f^{(m)}(r) \neq 0$. So the Taylor formula gives $f(r+h) = f(r) + hf'(r) + \cdots + [h^{m-1}/(m-1)!]f^{(m-1)}(r) + [h^m/m!]f^{(m)}(r) + \cdots$. Then $f(x_n) = f(r+e_n) = [e_n^m/m!]f^{(m)}(r) + e_n^{m+1}A$ where $A \equiv f^{(m+1)}(\xi_n)/(m+1)!$.

Similarly, $f'(x_n) = [e_n^{m-1}/(m-1)!]f^{(m)}(r) + e_n^mB$ where $B \equiv f^{(m+1)}(\eta_n)/m!$.

Now $e_{n+1} = x_{n+1} - r = x_n - r - mf(x_n)/f'(x_n) = e_n - mf(x_n)/f'(x_n) = [e_nf'(x_n) - mf(x_n)]/f'(x_n)$
$= \{e_n[e_n^{m-1}f^{(m)}(r)/(m-1)! + e_n^mB] - m[e_n^mf^{(m)}(r)/m! + e^{m+1})A]\}/ \{e_n^{m-1}f^{(m)}(r)/(m-1)! + e_n^mB\}$
$= [e_n^{m+1}B - me_n^{m+1}A]/[e^{m-1}f^{(m)}(r)/(m-1)! + e_n^mB] = e_n^2\{(B-mA)/[f^{(m)}(r)/(m-1)! + e_nB]$.
We need to assume $f, f', \ldots, f^{(m+1)}$ are continuous and that $f^{(m+1)}(r) \neq 0$.

23. **a.** $J = \begin{bmatrix} 8x_1 & -2x_2 \\ 4x_2^2 - 1 & 8x_1x_1 \end{bmatrix}$. So $J(0,1) = \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}$ and $J^{-1}(0,1) = (1/6)\begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$.

Thus, $\begin{bmatrix} h_1^{(1)} \\ h_2^{(1)} \end{bmatrix} = -(1/6)\begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/2 \end{bmatrix}$. So $\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix}$.

Next $\begin{bmatrix} h_1^{(2)} \\ h_2^{(2)} \end{bmatrix} = -(1/6)\begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 1/3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/6 \end{bmatrix}$.

Thus, $\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/2 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix}$.

§3.3: Question 7

7. $x_{n+1} = x_n - f(x_n)[(x_n - x_{n-1})/(f(x_n) - f(x_{n-1}))]$
$= [x_nf(x_n) - x_nf(x_{n-1}) - x_nf(x_n) + x_{n-1}f(x_n)]/[f(x_n) - f(x_{n-1})]$
$= [f(x_n)x_{n-1} - x_nf(x_{n-1})]/[f(x_n) - f(x_{n-1})]$. This is inferior to Eqn. (3) because as $x_n \to x_{n+1} \approx r$, $f(x_n) \to f(x_{n+1}) \approx f(r)$ resulting in $\approx [f(r)r - rf(r)]/[f(r) - f(r)]$ "catastrophic cancellation" with a loss of precision while Eqn. (3) produces $\approx r - f(r)[(r-r)/(f(r) - f(r))] \approx r$.