MATH 170B HOMEWORK 2 SOLUTIONS

§1.3: Questions 11, 14, 27

11. **a.** Characteristic equation: $x^3 - 3x^2 + 4 = 0$. Roots: $-1, 2$ (double).
    Basis: $[-1, 1, -1, 1, \ldots, (-1)^n, \ldots]$, $[2, 4, 8, 16, \ldots, 2^n, \ldots, \ldots]$, $[1, 4, 12, 32, \ldots, n2^{n-1}, \ldots]$.

    b. Characteristic equation: $3x^2 - 2x + 3 = 0$. Roots: $1 \pm i\sqrt{2}$.
    Basis: $u_n = (1 + i\sqrt{2}\,)^n$, $v_n = (1 - i\sqrt{2}\,)^n$.

    c. Characteristic equation: $2x^6 - 9x^5 + 12x^4 - 4x^3 = 0$. Roots: 0 (triple), 1/2 (simple), 2 (double).
    Basis: $x^{(1)} = [1, 0, 0, \ldots]$, $x^{(2)} = [0, 1, 0, \ldots]$, $x^{(3)} = [0, 0, 1, \ldots]$, $x^{(4)} = [1/2, 1/4, 1/8, \ldots]$,
    $x^{(5)} = [2, 4, 8, \ldots]$, $x^{(6)} = [1, 4, 12, \ldots]$. Here the general term for $x^{(6)}$ is $x_n^{(6)} = n2^{n-1}$.

    d. Characteristic equation: $\pi x^2 - \sqrt{2}x + \log 2 = 0$. Roots: $x_1, x_2 = (\sqrt{2} \pm i\sqrt{4\pi \log 2 - 2})/(2\pi)$.
    Basis: $u_n = x_1^n$, $v_n = x_2^n$.

14. It is obvious that $\Delta = E - I$. If p is a polynomial of degree n, then by Taylor's Theorem $p(x) = \sum_{j=0}^{n}[p^{(j)}(a)/j!](x-a)^j$. Put $x = E$ and $a = I$ to get $p(E) = \sum_{j=0}^{n}(1/j!)p^{(j)}(I)\Delta^j$.
    $\sum_j c_j E^{j+1} = \left(\sum_j c_j E^j\right) E$. So $p(E)Ex = 0$; i.e., Ex is a solution.

27. Characteristic equation: $\lambda^2 - 2\lambda - 2 = 0$. Roots: $1 \pm \sqrt{3}$.
    General solution: $z_n = \alpha(1 + \sqrt{3}\,)^n + \beta(1 - \sqrt{3}\,)^n$.
    Initial values give $1 = x_1 = \alpha(1 + \sqrt{3}\,) + \beta(1 - \sqrt{3}\,)$ and $1 - \sqrt{3} = x_2 = \alpha(1 + \sqrt{3}\,)^2 + \beta(1 - \sqrt{3}\,)^2$.
    So solution is $\alpha = 0$ and $\beta = 1/(1 - \sqrt{3}\,)$.

§2.3: Question 8, 9

8. Assume $r_n = \lambda^n$ gives characteristic equation $\lambda^2 - \lambda - 1 = 0$ with roots $\lambda_1 = [(1 + \sqrt{5}\,)/2]$ and $\lambda_2 = [(1 - \sqrt{5}\,)/2]$. General solution: $r_n = A\lambda_1^n + B\lambda_2^n$ where $A = [(1 + \sqrt{5}\,)/2]/\sqrt{5}$ and
   $B = [(1 - \sqrt{5}\,)/2]/\sqrt{5}$. Now $r_n/r_{n-1} = (A\lambda_1^n + B\lambda_2^n)/(A\lambda_1^{n-1} + B\lambda_2^{n-1})$
   $= \lambda_1[A + B\theta^n]/[A + B\theta^{n+1}] \to \lambda_1$ since $\theta = \lambda_2/\lambda_1 <$. The convergence has linear behavior.

9. As above, $r_n = A\lambda_1^n + B\lambda_2^n = A[(1 + \sqrt{5}\,)/2]^n + B[(1 - \sqrt{5}\,)/2]^n$. Since the root $|\lambda_1| > 1$, the recurrence relation does not provide a stable means for computing r_n. In this case, $A = 0$ and $B = 1$ so $r_n = [(1 - \sqrt{5}\,)/2]^n \to 0$ as $n \to \infty$.

§3.4: Question 12, 13

12. $x = \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}$ Let $x_1 = \sqrt{p}$, $x_2 = \sqrt{p + \sqrt{p}}$, $x_3 = \sqrt{p + \sqrt{p + \sqrt{p}}}$, and so on. Observe that $x_2 = \sqrt{p + x_1}$, $x_3 = \sqrt{p + x_2}$, and so on. In general $x_{n+1} = \sqrt{p + x_n}$ (I). Let $f(x) = \sqrt{p + x}$. Equation (I) is the result of using functional iteration on f. If $\lim x_n$ exists, denote it by x. Take limits in Equation (I) to get $x = \sqrt{p + x}$. Hence, $x^2 = p + x$, $x^2 - x - p = 0$, $x = (1 + \sqrt{1 + 4p})/2$. This is the limit of the sequence. For example if $p = 2$, $x = 2$. Try it on your pocket calculator.

13. Use the ideas of Problem 3.4.12. Let $x_1 = 1/p$, $x_2 = 1/(p + (1/p))$, $x_3 = 1/(p + (1/p + (1/p)))$ etc. So $x_2 = 1/(p + x_1)$, $x_3 = 1/(p + x_2)$, and so on. Hence, $x_{n+1} = 1/(p + x_n)$. If $\lim_{n\to\infty} x_n = x$ then $x = 1/(p + x)$. Hence, $x(p + x) = 1$, $x^2 + px - 1 = 0$, $x = (-p + \sqrt{p^2 + 4})/2$. This illustrates functional iteration with $f(x) = 1/(p + x)$. If $p > 1$, f is a contraction. Use Mean Value Theorem: $|f(x) - f(y)| = |f'(\xi)|\,|x - y| = |-1/(p + \xi)^2|\,|x - y|$. Since $p > 1$, all x_n's will be ≥ 0, and $1/(p + x)^2 \leq 1/p^2 < 1$. So f is a contraction on $[0, \infty]$. f actually maps $[0, 1]$ into $[0, 1]$, so has a fixed point in $[0, 1]$.