MATH 170B HOMEWORK 4 SOLUTIONS

§6.4: Questions 5, 7, 9

5. $f(1^-) = 1 = f(1^+)$, so f is continuous at $x = 1$. Also $f(2^-) = 3/2 = f(2^+)$, so f is continuous at $x = 2$. $f'(x) = \begin{cases} 1 & x \in (-\infty, 1] \\ 2 - x & x \in [1, 2] \\ 0 & x \in [2, \infty) \end{cases}$.

   Thus, $f'(1^-) = 1 = f'(1^+)$ and $f'(2^-) = 0 = f'(2^+)$. Therefore, $f'(x)$ is continuous at $x = 1$, $x = 2$. Hence, f is a quadratic spline function.

7. Enforce the continuity of f at knots: 1,3. At $x = 1$, $a(-1)^2 + 0 = c(-1)^2 \Rightarrow a = c$. At $x = 3$, $c(1)^2 = d(1)^2 + 0 \Rightarrow c = d$. Continuity of f' at knots: At $x = 1$, $2a(-1) + 0 = 2c(-1) \Rightarrow a = c$. At $x = 3$, $2c = 2d + 0 \Rightarrow c = d$. Continuity of f'' at knots: At $x = 1$, $2a + 0 = 2c \Rightarrow a = c$. At $x = 3$, $2c = 2d + 0 \Rightarrow c = d$. Thus, in order that f be a cubic spline: $a = c = d$ and b, e any arbitrary values. Next, determine a, b, c, d, e so that f interpolates the table. At $x = 0$, $a(-2)^2 + b(-1)^3 = 26 \Rightarrow 4a - b = 26$. At $x = 1$, $a(-1)^2 + b \cdot 0 = 7 \Rightarrow a = 7$. So $b = 2$ and $c = d = 7$. At $x = 4$, $d(2)^2 + e(1)^3 = 25 \Rightarrow 28 + e = 25 \Rightarrow e = -3$. Then: $a = c = d = 7$, $b = 2$, $e = -3$.

9. Put $q_i(x) = \frac{1}{2}(z_{i+1}/h_i)(x - t_i)^2 - \frac{1}{2}(z_i/h_i)(t_{i+1} - x)^2 + y_i + \frac{1}{2}z_i h_i$.
   Then $q_i(t_i) = -\frac{1}{2}(z_i/h_i)(t_{i+1} - t_i)^2 + y_i + \frac{1}{2}(z_i h_i) = \frac{1}{2}[-(z_i h_i^2)/h_i] + \frac{1}{2}(z_i h_i) + y_i = y_i$ where $h_i = t_{i+1} - t_i$. $q_i'(x) = (z_{i+1}/h_i)(x - t_i) + (z_i/h_i)(t_{i+1} - x)$, $q_i'(t_i) = (z_i/h_i)(t_{i+1} - t_i) = z_i$, $q_i'(t_{i+1}) = (z_{i+1}/h_i)(t_{i+1} - t_i) = z_{i+1}$. $q_{i-1}(x) = \frac{1}{2}(z_i/h_{i-1})(x - t_{i-1})^2 - \frac{1}{2}(z_{i-1}/h_{i-1})(t_i - x)^2 + y_{i-1} + \frac{1}{2}(z_{i-1}h_{i-1})$ $q_{i-1}(t_i) = \frac{1}{2}(z_i/h_{i-1})(t_i - t_{i-1})^2 + y_{i-1} + \frac{1}{2}(z_{i-1}h_{i-1}) = \frac{1}{2}(z_i + z_{i-1})h_{i-1} + y_{i-1}$.
   Continuity Condition $\frac{1}{2}(z_i + z_{i-1})h_{i-1} + y_{i-1} = y_i \Rightarrow z_i + z_{i-1} = (2/h_{i-1})(y_i - y_{i-1})$. $q_i(x) = \frac{1}{2}(z_{i+1}/h_i)(x - t_i)^2 - \frac{1}{2}(z_i/h_i)(t_{i+1} - x)^2 + y_i + \frac{1}{2}(z_i h_i) = \frac{1}{2}(z_{i+1}/h_i)(x - t_i)^2 - \frac{1}{2}(z_i/h_i)(x - t_i - h_i)^2 + y_i + \frac{1}{2}(z_i h_i) = \frac{1}{2}[(z_{i+1} - z_i)/h_i](x - t_i)^2 + z_i(x - t_i) - \frac{1}{2}(z_i h_i) + y_i + \frac{1}{2}(z_i h_i) = \frac{1}{2}[(z_{i+1} - z_i)/h_i](x - t_i)^2 + z_i(x - t_i) + y_i$.

   Here $i = 1, 2, \ldots, n-1$. Q: piecewise quadratic Q, Q' continuous $Q'(t_i) = z_i$ well-defined
   $q_1(t_2) = q_2(t_2)$ etc. $q_{n-2}(t_{n-1}) = q_{n-1}(t_{n-1})$, i.e., $q_{i-1}(t_i) = q_i(t_i)$ for $i = 2 \ldots n-1$ $z_{i-1} + z_i = (2/h_{i-1})(y_i - y_{i-1})$ $(2 \le i \le n-1)$.
   Let $z_1 = 0$ and define inductively $z_i = (2/h_{i-1})(y_i - y_{i-1}) - z_{i-1}$, $i = 2, 3, \ldots, n$. z_i is arbitrary, $z_i = (2/h_{i-1})(y_i - y_{i-1}) - z_{i-1}$ $i = 2, \ldots, n$.
   So $z_i = \alpha_i - z_{i-1}$. $z_2 = \alpha_2 - z_1$, $z_3 = \alpha_3 - z_2 = \alpha_3 - (\alpha_2 - z_1) = \alpha_3 - \alpha_2 + z_1$, $z_4 = \alpha_4 - z_3 = \alpha_4 - \alpha_3 + \alpha_2 - z_1$, Etc.. $z_1 = \alpha_1 - \alpha_{i-1} + \alpha_{i-2} \cdots + (-1)^i(\alpha_2 - z_1)$.

   So $z_i = \gamma_i - (-1)^i z_1$, $\gamma_2 = \alpha_2$, $\gamma_3 = \alpha_3 - \gamma_2$, $\gamma_4 = \alpha_3 - \gamma_3$, etc. $\Phi = \sum_{i=2}^n z_i 62 = z_2^2 + z_3^2 + z_4^2 + \cdots + z_n^2$ $= (\gamma_2 - z_i)^2 + (\gamma_3 + z_1)^2 + (\gamma_4 - z_1)^2 + \cdots + (\gamma_n - (-1)^n z_1)^2$ $d\Phi/dz_1 = -2(\gamma_2 - z_1) + 2(\gamma_3 + z_1) - 2(\gamma_4 - z_1) - \cdots - 2(-1)^n(\gamma_n - (-1)^n z_1) = 0$ $-\underbrace{\gamma_2 + z_1}_{1} + \underbrace{\gamma_3 + z_1}_{2} - \underbrace{\gamma_4 + z_1}_{3} \cdots - (-1)^n \underbrace{\gamma_n + z_1}_{n-1} = 0$
   $(n-1)z_1 - (\gamma_2 - \gamma_3 + \gamma_4 - \gamma_5 + \cdots + (-1)^n\gamma_n) = 0$ $z_1 = (\gamma_2 - \gamma_3 + \gamma_4 - \gamma_4 + \cdots + (-1)^n\gamma_n)/(n-1)$.

   Now $\gamma_2 - \gamma_3 = \alpha_2 - (\alpha_3 - \alpha_2) = 2\alpha_2 - \alpha_3$ and $\gamma_4 - \gamma_5 = \gamma_4 - (\alpha_4 - \gamma_4) = 2\gamma_4 - \alpha_4$. $\gamma_2 = \alpha_2$, $\gamma_3 = \alpha_3 - \alpha_2$, $\gamma_4 = \alpha_4 - \alpha_3 + \alpha_2$, $\gamma_5 = \alpha_5 - \alpha_4 + \alpha_3 - \alpha_2$, etc. $\gamma_2 = \alpha_2$, $\gamma_3 = \alpha_2 - \alpha_3$, $\gamma_4 = \alpha_2 - \alpha_3 + \alpha_4$, $-\gamma_5 = \alpha_2 - \alpha_3 + \alpha_4 - \alpha_5$, etc. So $[\gamma_2 - \gamma_3 + \gamma_4 - \gamma_5 + \cdots + (-1)^n\gamma_n]/(n-1)$ $= [(n-1)\alpha_2 - (n-2)\alpha_3 + (n-3)\alpha_4 - \cdots]/(n-1)$.