MATH 170B HOMEWORK 6 SOLUTIONS

§6.8: Questions 3, 5, 9, 20, 23

3. The general result is as follows. Let f be a continuous even function on the interval $[-a, a]$. Let G be a subspace of $C[-a, a]$ with the property that if $g \in G$ then the function $x \mapsto g(-x)$ also belongs to G. If g minimizes the integral $\int_{-a}^{a} [f(x) - g(x)]^2\, dx$, then g is even. To prove this, let $h(x) = g(-x)$. By changing the variable from x to $-y$ in the integral $\int_{-a}^{a} [f(x) - h(x)]^2\, dx$ we see that h also minimizes the integral. By Theorem 2, $f - g \perp G$ and $f - h \perp G$. Hence $(f - g) - (f - h) \perp G$. In particular $h - g \perp h - g$, and so $h = g$. Thus g is even.

5. Let $f = \sum_i a_i u_i$ and $g = \sum_j b_j u_j$. Then $\langle f, g\rangle = \langle \sum_i a_i u_i, \sum_j b_j u_j\rangle = \sum_i \sum_j a_i b_j \langle u_i, u_j\rangle = \sum_i a_i b_i = \sum_i \langle f, u_i\rangle\langle g, u_i\rangle$.

9. Let $g = \sum_j c_j v_j$. By Theorem 2, g will be the best approximation to f if $f - g \perp v_i$ for $i = 1, \ldots, n$. Thus $\langle f, v_i\rangle = \langle g, v_i\rangle = \langle \sum_j c_j v_j, v_i\rangle = c_i\langle v_i, v_i\rangle$ by orthogonality. If $v_i \neq 0$, let $c_i = \langle f, v_i\rangle / \langle v_i, v_i\rangle$. If $v_i = 0$, c_i can be arbitrarily chosen.

20. The best approximation of f is $g = \sum_i \langle f, g_i\rangle g_i$, by Theorem 3. Hence $\mathrm{dist}(f, G) = \|f - g\|$. Now $\|f - g\|^2 = \|f\|^2 - 2\langle f, g\rangle + \|g\|^2 = \|f\|^2 - 2\sum_i \langle f, g_i\rangle^2 + \sum_i \langle f, g_i\rangle^2 = \|f\|^2 - \sum_i \langle f, g_i\rangle^2$. Here we used Lemma 2 and Part (b) of Lemma 1. Thus $\mathrm{dist}(f, G) = \{\|f\|^2 - \sum_i \langle f, g_i\rangle^2\}^{1/2}$.