Assigned: Monday, February 18, 2013
Due Date: Friday, March 1, 2013

MATH 170B: PROJECT 2
"NONLINEAR SYSTEMS & POLYNOMIAL INTERPOLATION"
WINTER 2013

Nonlinear Systems

1. $\boxed{\textbf{SOURCE CODE:}}$

   Write a MATLAB function for **Newton's method for systems**. Write this function so that it works for a 2×2 system (in other words: you don't have to write to solve a general $m \times m$ system). Make use of the 2×2 inversion formula to invert the necessary linear system in each Newton iteration. Your function should be written so that it can be called in MATLAB by typing:

   - `[x,NumIters] = NewtSys2x2(@F,@J,x0,TOL,MaxIters)`

   Here $F(x_1, x_2)$ is the vector-valued function whose root we are trying to approximate and $J(x_1, x_2)$ is its Jacobian matrix. Include your source code with the rest of your assignment.

2. $\boxed{\textbf{APPLICATION PROBLEM:}}$
   Let $i = \sqrt{-1}$. Find all five roots of the complex polynomial

   $$f(z) = (1+i)z^5 - 2z^3 + iz^2 - 1$$

   to within a tolerance of 10^{-7} using your **Newtons' method for systems** code. For each root report your intial guess, how many iterations were required for a 10^{-7} error, and your approximate root.

   HINT: The roots of $f(z)$ are the points z in the complex plane where $f(z) = 0$. We can write $z = x + iy$ where x is the real part and y is the imaginary part. In this form, both x and y are strictly real numbers.

   We can also separate $f(z)$ into a real part and an imaginary part:

   $$\begin{aligned} f(z) &= (1+i)(x+iy)^5 - 2(x+iy)^3 + i(x+iy)^2 - 1 \\ &. \\ &. \\ \Longrightarrow \quad f(z) &= A(x,y) + iB(x,y)\,. \end{aligned}$$

   where

   $$A(x,y) = x^5 - 5x^4y - 10x^3y^2 + 10x^2y^3 + 5xy^4 - y^5 - 2x^3 + 6xy^2 - 2xy - 1$$

   $$B(x,y) = x^5 + 5x^4y - 10x^3y^2 - 10x^2y^3 + 5xy^4 + y^5 - 6x^2y + 2y^3 + x^2 - y^2.$$

Again, x and y are strictly real numbers and $A(x,y)$ and $B(x,y)$ are strictly real functions.

The roots of $f(z)$ are all the points z such that $f(z)=0$. Using the above result this is equivalent to saying that the roots of $f(z)$ are all the points (x,y) such that $A(x,y)=B(x,y)=0$.

Polynomial Interpolation

3. Consider the function
$$f(x) = \sin\left(\frac{\pi}{2}x\right) + \frac{x}{2}\,.$$
   (a) We would like to construct a polynomial $P(x)$ that interpolates $f(x)$ at the points $x_0 = 0$, $x_1 = 2$, and $x_2 = 3$. Compute all the necessary Lagrange polynomials and use these to construct the interpolating polynomial $P(x)$.
   (b) Compute the error term $R(x)$.
   (c) **Interpolation:** Compute the maximum bound on $R(x)$ if we approximate $f(1)$ by $P(1)$.
   (d) **Extrapolation:** Compute the maximum bound on $R(x)$ if we approximate $f(4)$ by $P(4)$.

4. $\boxed{\textbf{SOURCE CODE:}}$

   Let $x_0, x_1, \ldots, x_n$ be distinct points with corresponding data values $f_0, f_1, \ldots, f_n$. Let $P(x)$ be the unique polynomial of degree n which interpolates the data (i.e., $P(x_i) = f_i$ for each $i = 0, \ldots, n$). Write the following MATLAB routines:

   (a) `F = DivDiff(x,f,n)` – a **function** to compute the divided differences.
   (b) `P = Horners(x,F,xbar)` – a **function** which uses the divided differences to evaluate $P(x)$ at the point `xbar` using Horner's method.

5. Consider the function

$$f(x) = \frac{1}{1 + 25x^2}\,.$$

   (a) Let x_i be 11 equally spaced points on $[-1, 1]$:

$$x_i = -1 + i/5, \qquad \text{where} \quad i = 0, 1, 2, \ldots, 10.$$

      i. Use your code from the previous problem to construct the divided differences with these x_i's.

      ii. Use your Horner's method code to evaluate the interpolating polynomial $P(x)$ at all the points z_k:

```
z = linspace(-1,1,500);
for k=1:500
   P(k) = Horners(x,F,z(k));
end
```

      iii. Make a single MATLAB plot that contains: (z, P), $(z, f(z))$, and the data points $(x_i, f(x_i))$ for $i = 0, 1, 2, \ldots 10$. Make the $P(z)$ plot a solid line (`'b-'`), the $f(z)$ a dashed line (`'r--'`), and the data points stars (`'k*'`).

   (b) Now, let x_i be 11 *unequally* spaced points on $[-1, 1]$:

$$x_i = \cos\left[\frac{\pi\,(i + 0.5)}{11}\right], \qquad \text{where} \quad i = 0, 1, 2, \ldots, 10.$$

   Repeat steps (i), (ii), and (iii) for these unequally spaced data points.

   (c) Explain what you observe with these two choices of the x_i's.