1. The function $f(x) = \cos(x)$ has a root at $x = \pi/2$. Using the theory we developed for fixed point iterations, find the largest interval around $x = \pi/2$ in which we can choose an initial guess for Newton’s method and still be guaranteed to converge to $\pi/2$.

2. Show that the iteration scheme

$$\alpha_{n+1} = g(\alpha_n) = \frac{\alpha_n^2 - a\alpha_n + a^2 + 5a}{\alpha_n + 5}, \quad n \geq 0$$

converges to the fixed point $a$ quadratically (i.e., order of convergence is 2) for all $a \neq -5$.

(HINT: subtract $a$ from both sides, manipulate the expression so that it has the same form as in the definition of order of convergence, then take the limit as $n \to \infty$. Note that $\alpha_n \to a$ as $n \to \infty$. This problem is very similar to Problem #4 of Project #1.)

3. Consider the following 2 matrices,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$  

Find the conditions that must be satisfied by $a$, $b$, $c$, and $d$ so that $AB = BA$.

4. Consider the following linear system,

$$
\begin{align*}
    x_1 - x_2 + \alpha x_3 &= -2 \\
    -x_1 + 2x_2 - \alpha x_3 &= 3 \\
    \alpha x_1 + x_2 + x_3 &= 2.
\end{align*}
$$

(a) Write this linear system of equations in matrix-vector notation.

(b) Reduce the augmented matrix to an upper triangular system via Gaussian elimination (Show EACH STEP in the reduction).

(c) Using the result in part (b), find values of $\alpha$ for which the system has no solutions.

(d) Using the result in part (b), find values of $\alpha$ for which the system has infinitely many solutions.

(e) Assuming a unique solution exists for a given $\alpha$, use back substitution to solve for $x_1$, $x_2$, and $x_3$.
5. [taken from Trefethen and Bau (1997)] Let $B$ be a $4 \times 4$ matrix to which we apply the following operations:

- double column 1,
- halve row 3,
- add row 3 to row 1,
- interchange columns 1 and 4,
- subtract row 2 from each of the other rows,
- replace column 4 by column 3,
- delete column 1 (so that the column dimension is reduced by 1).

(a) Write the results as a product of eight matrices.
(b) Write it again as a product $ABC$ (same $B$) of three matrices.