MATH 174/274: Homework VII
“Splines and Numerical Integration”
Fall 2015

NOTE: For each homework assignment observe the following guidelines:

1. [Taken from Burden & Faires, 5th edition, 1993]
   A natural cubic spline $S$ on $[0, 2]$ is defined by
   \[
   S(x) = \begin{cases} 
   S_0(x) = 1 + 2x - x^3 & \text{if } 0 \leq x < 1 \\
   S_1(x) = 2 + b(x - 1) + c(x - 1)^2 + d(x - 1)^3 & \text{if } 1 \leq x \leq 2 
   \end{cases}
   \]
   Find $b$, $c$, and $d$.

2. [Taken from Burden & Faires, 5th edition, 1993]
   A clamped cubic spline $S$ for a function $f(x)$ on $[1, 3]$ is defined by
   \[
   S(x) = \begin{cases} 
   S_0(x) = 3(x - 1) + 2(x - 1)^2 - (x - 1)^3 & \text{if } 1 \leq x < 2 \\
   S_1(x) = a + b(x - 2) + c(x - 2)^2 + d(x - 2)^3 & \text{if } 2 \leq x \leq 3 
   \end{cases}
   \]
   Given that $f'(1) = f'(3)$, find $a$, $b$, $c$, and $d$.

3. Consider the following 4 equally spaced points on the interval $[x_0, x_3]$:
   \[x_j = x_0 + jh \quad [j = 0, 1, 2, 3],\]
   where $h = (x_3 - x_0)/3$.
   (a) Construct all the Lagrange polynomials $L_{3,j}(x)$ that correspond to the points $x_0$, $x_1$, $x_2$, and $x_3$.
   (b) Use these Lagrange polynomials to construct the interpolating polynomial $P_3(x)$ that interpolates the function $f(x)$ at the points $x_0$, $x_1$, $x_2$, and $x_3$.
   (c) Integrate the interpolating polynomial $P_3(x)$ to derive the following Newton-Cotes formula often referred to as Simpson’s three-eighths rule:
   \[
   \int_{x_0}^{x_3} f(x) \, dx \approx \frac{3h}{8} \left[ f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right].
   \]

4. Consider the following integral:
   \[
   \int_{3}^{4} \frac{x}{\sqrt{x^2 - 4}} \, dx.
   \]
   Compute this integral:
   (a) Exactly by hand.
   (b) Approximately with the Trapezoidal rule and compute the error (error = |approx - exact|).
   (c) Approximately with Simpson’s rule and compute the error (error = |approx - exact|).
   (d) Approximately with Simpson’s three-eighths rule and compute the error (error = |approx - exact|).