MATH 174/274: Homework 1  
"Taylor Series and Round-Off Error"  
Fall 2011

1. Compute the first 5 terms in the Taylor series (constant, linear, quadratic, cubic, and quartic pieces) for the following functions:
   (a) \( f(x) = \sqrt{4x - x^2} \), about the point \( x = 2 \)
   (b) \( f(x) = 3 \tan(x) \), about the point \( x = \pi/4 \)

2. Using the results from Problem 1(a), make a single MATLAB plot which contains all of the following:
   (a) a graph of \( f(x) = \sqrt{4x - x^2} \) versus \( x \) for \( x \in (0, 4) \).
   (b) a graph of \( p_0(x) \).
   (c) a graph of \( p_2(x) \).
   (d) a graph of \( p_4(x) \).
   (e) a title, \( x \)-axis label, \( y \)-axis label, and a legend.

3. The quotient
   \[ g(x) = \frac{\log(1 + xe^x)}{x} \]

   seems at first glance to be undefined at \( x = 0 \). Approximate \( \log(1 + xe^x) \) by a Taylor polynomial of degree 2 about the point \( x = 0 \). Use this Taylor approximation to determine a natural definition of \( g(0) \).

4. Consider the function \( f(x) = 1/(1 - x) \).
   (a) Obtain the infinite Taylor series representation of \( f(x) \) about the point \( x = 0 \).
   (b) Obtain the infinite Taylor series representation of \( g(t) = 1/(1 + t^2) \) about \( t = 0 \).
      Do this by substituting \( x = -t^2 \) into the Taylor expansion of \( f(x) \).
   (c) Obtain the infinite Taylor series representation of \( \tan^{-1}(x) \) about \( x = 0 \). Do this by integrating the result in (b):
      \[ \tan^{-1}(x) = \int_0^x \frac{1}{1+t^2} \, dt \].

5. Consider the function
   \[ g(x) = e^x. \]
   (a) Derive the general \( n^{th} \) order Taylor polynomial \( (p_n(x)) \) as well the remainder term \( (R_n(x)) \) for the function \( g(x) \) expanded about the point \( x = 0 \).
(b) Using the remainder term from part (a), determine the value of $n$ needed to guarantee that $|p_{n}(1) - g(1)| < 10^{-5}$.

6. Consider the following:

$$x_n = 1 - \cos \left( \frac{\pi}{n} \right)$$

$$y_n = 2 \sin^2 \left( \frac{\pi}{2n} \right)$$

**NOTE:** From trigonometry we know that $x_n = y_n$ since $\sin^2(\theta) = \frac{1}{2} (1 - \cos(2\theta))$.

(a) Subplot #1: Plot $x_n$ and $y_n$ versus $n$ for $n = 1, \ldots, 1000$ on a single log-log plot using the `loglog` command. Use a solid line for $x_n$ and open circles for $y_n$.

(b) Subplot #2: Plot the relative error $|x_n - y_n|/|y_n|$ on a log-log plot using the `loglog` command.

(c) As $n$ becomes large, which one ($x_n$ or $y_n$) is more accurate? Why? (HINT: recall the 2 situations that you want to avoid when doing computer arithmetic.)