MATH 179/279 HOMEWORK 6

Due: Thursday, May 23, 2019.

(1) The heat equation with no source term is given by,

\[ u_t - \kappa u_{xx} = 0, \]

with initial condition \( u(x, 0) = u_0(x) = 0 \) and Dirichlet boundary conditions, \( u(0, t) = u_s \) and \( u(L, t) = 0 \). The exact solution is given by,

\[ u(x, t) = \left(1 - \frac{x}{L}\right) u_s + \sum_{k \in \mathbb{N}^*} \left( -\frac{2u_s}{k\pi} \right) \exp \left( -\left(\frac{k\pi}{L}\right)^2 \kappa t \right) \sin \left( \frac{k\pi}{L} x \right). \]

Consider the following explicit centered scheme for the heat equation:

\[ \frac{u_{j,n+1} - u_{j,n}}{\Delta t} - \kappa \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{\Delta x^2} = 0. \]

The stability condition for this scheme is,

\[ \kappa \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}. \]

(a) Write a program to solve the problem of heat propagation, setting \( \kappa = 1, \ L = 1, \ u_s = 1 \), and let \( \Delta x = L/50 \). Use the stability condition to choose the time step \( \Delta t \).

(b) Plot the numerical solution for different times and compare it to the exact solution given above, where the exact solution is approximated by considering the first 20 wave numbers \( k \) in the expansion.

(c) Comment on the results for small \( t \) and for large \( t \).

(d) Now run the program with \( u_s = 0 \), and initial condition given by,

\[ u_0(x) = u(x, 0) = \sin \left( \frac{\pi}{L} x \right) + \frac{1}{4} \sin \left( 10 \frac{\pi}{L} x \right). \]

(e) Compare the numerical solution to the exact solution for zero initial data, and comment on the damping of the two waves in the initial condition.