MATH 179/279 HOMEWORK 6

Due: Thursday, May 23, 2019.

(1) The heat equation with no source term is given by,

$$u_t - \kappa u_{xx} = 0,$$

with initial condition $u(x,0) = u_0(x) = 0$ and Dirichlet boundary conditions, $u(0,t) = u_s$ and $u(L,t) = 0$. The exact solution is given by,

$$u(x,t) = \left(1 - \frac{x}{L}\right) u_s + \sum_{k \in \mathbb{N}^*} \left(-\frac{2u_s}{k\pi}\right) \exp\left(-\left(\frac{k\pi}{L}\right)^2 \kappa t\right) \sin\left(\frac{k\pi}{L} x\right).$$

Consider the following explicit centered scheme for the heat equation:

$$\frac{u_{j,n+1} - u_{j,n}}{\Delta t} - \kappa \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{\Delta x^2} = 0.$$

The stability condition for this scheme is,

$$\kappa \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}.$$

(a) Write a program to solve the problem of heat propagation, setting $\kappa = 1$, $L = 1$, $u_s = 1$, and let $\Delta x = L/50$. Use the stability condition to choose the time step Δt.

(b) Plot the numerical solution for different times and compare it to the exact solution given above, where the exact solution is approximated by considering the first 20 wave numbers k in the expansion.

(c) Comment on the results for small t and for large t.

(d) Now run the program with $u_s = 0$, and initial condition given by,

$$u_0(x) = u(x,0) = \sin\left(\frac{\pi}{L} x\right) + \frac{1}{4} \sin\left(10 \frac{\pi}{L} x\right).$$

(e) Compare the numerical solution to the exact solution for zero initial data, and comment on the damping of the two waves in the initial condition.