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Attitude Maneuvers of a Rigid Spacecraft in a Circular Orbit

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***Abstract*— A global model is presented that can be used to study attitude maneuvers of a rigid spacecraft in a circular orbit about a large central body. The model includes gravity gradient effects that arise from the non-uniform gravity field and characterizes the spacecraft attitude with respect to the uniformly rotating local vertical local horizontal coordinate frame. An accurate computational approach for solving a nonlinear boundary value problem is proposed, assuming that control torque impulses can be applied at initiation and at termination of the maneuver. If the terminal attitude condition is relaxed, then an accurate computational approach for solving the minimal impulse optimal control problem is presented. Since the attitude is represented by a rotation matrix, this approach avoids any singularity or ambiguity arising from other attitude representations such as Euler angles or quaternions.**

I. INTRODUCTION

The attitude dynamics of an uncontrolled rigid spacecraft in a circular orbit about a large central body, including gravity gradient effects, have been extensively studied; see [1], [2]. There are 24 distinct relative equilibria for which the principal axes are exactly aligned with the local vertical local horizontal (LVLH) axes, and the spacecraft angular velocity is identical to the orbital angular velocity of the LVLH coordinate frame. Linear rotational equations of motion that describe small perturbations from any relative equilibrium solutions are well known. Linear attitude control of a rigid spacecraft in a circular orbit, including linear gravity gradient effects, has also been addressed in [2]. However, linear controllers have the limitation that they are only applicable to small attitude change maneuvers.

The emphasis in this paper is on large angle attitude maneuvers of a rigid spacecraft in a circular orbit about a large central body, including gravity gradient effects. A nonlinear, globally defined model is introduced. This model describes the attitude of the spacecraft, relative to the uniformly rotating LVLH coordinate frame, by a rotation matrix. The model includes gravity gradient terms that reflect the rotation of the LVLH frame, and terms that reflect control input torques. In the problems studied in this paper, independent impulsive control torques can be applied about each principal axis. Gravity gradient moments are significant in Earth orbits for orbit altitudes between $400\,\mathrm{km}$ and $40,000\,\mathrm{km}$ and for attitude maneuver times that are not small compared with the orbital period.

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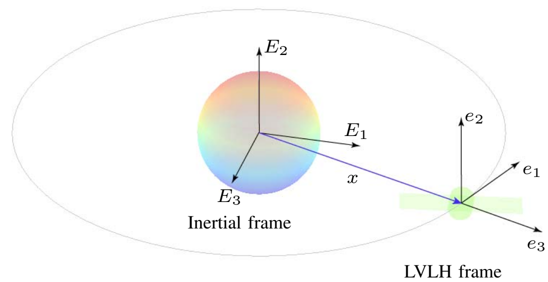


Fig. 1. Coordinate frames

We study open loop attitude maneuvers that can be accomplished by using two impulsive torque controls, one occurring at the initial time and one occurring at the terminal time of the maneuver. The attitude motion in between the initial time and the final time is uncontrolled. Two classes of attitude maneuver problems are studied. In Section III, a terminal attitude is specified so that there is a unique impulse sequence satisfying the boundary conditions. This problem can be solved computationally use a root finding algorithm. In Section IV, the specified terminal attitude condition is relaxed such that there are many impulse sequences that satisfy the boundary conditions. In this case we seek the minimum total impulse sequence that satisfies the boundary conditions. This problem can be solved computationally using a constrained minimization algorithm.

For both classes of attitude maneuver problems, our contribution is to demonstrate how sensitivity derivatives, used in the computational algorithms, can be determined effectively and accurately such that they satisfy the global geometry of the problem. Since the attitude is represented by a rotation matrix in the special orthogonal group $\mathrm{SO}(3)$, and the sensitivity derivatives are expressed in terms of the Lie algebra $\mathfrak{so}(3)$, this approach completely avoids singularities or ambiguities that arise from other representations of the rotation group, such as Euler angles and quaternions.

II. RIGID SPACECRAFT MODELS IN A CIRCULAR ORBIT

We assume that a rigid spacecraft is on a circular orbit with a constant orbital angular velocity $\omega_0 \in \mathbb{R}$. In this section, the continuous equations of motion and a geometric numerical integrator, referred to as Lie group variational integrator, for the attitude maneuver of a spacecraft in a circular orbit are given following the results of [3]. We identify $\mathrm{TSO}(3) \simeq$

$\mathrm{SO}(3)\times\mathfrak{so}(3)$ by left translation, and we identify $\mathfrak{so}(3)\simeq\mathbb{R}^3$ by an isomorphism $S(\cdot):\mathbb{R}^3\mapsto\mathfrak{so}(3)$.

Continuous equations of motion: We define three rotation matrices in SO(3);

R^{bi} : from the body fixed frame to the inertial frame,

R^{li} : from the LVLH frame to the inertial frame,

R^{bl} : from the body fixed frame to the LVLH frame,

where the inertial frame and the LVLH frame are illustrated in Fig. 1. Thus, $R^{bl}=R^{li^T}R^{bi}$.

The on-orbit spacecraft equations of motion are given by

$$\dot{\Pi}+\Omega\times\Pi=M^g, \quad (1)$$

$$\dot{R}^{bi}=R^{bi}S(\Omega), \quad (2)$$

$$\dot{R}^{li}=R^{li}S(\omega_0 e_2), \quad (3)$$

$$\dot{R}^{bl}=R^{bl}S(\Omega-\omega_0 R^{bl^T}e_2), \quad (4)$$

where $\Pi,\Omega\in\mathbb{R}^3$ are the angular momentum and the angular velocity of the spacecraft expressed in the body fixed frame, respectively, and $M^g\in\mathbb{R}^3$ is the gravity gradient moment. The isomorphism between $\mathbb{R}^3$ and $\mathfrak{so}(3)$ is defined such that $S(x)y=x\times y$ for any $x,y\in\mathbb{R}^3$. Since the orbital angular velocity ω_0 is constant, the solution of (3) is given by $R^{li}(t)=R^{li}(0)e^{S(\omega_0 e_2)t}$.

Gravity gradient moment: The gravity gradient moment is derived in [2]. We present an alternative way to obtain the gravity gradient moment directly using the gravity potential;

$$U=-\int_{\mathcal{B}}\frac{GM}{\|x+R^{bi}\rho\|}\,dm,$$

where $x\in\mathbb{R}^3$ is the position of the spacecraft in the inertial frame, and $\rho\in\mathbb{R}^3$ is a vector from the center of mass of the spacecraft to a mass element in the body fixed frame. G is the gravitational constant and M is the mass of the Earth.

From [3], the gravity gradient moment M^g can be determined by using the following relationship;

$$S(M^g)=\frac{\partial U}{\partial R^{bi}}^T R^{bi}-R^{bi^T}\frac{\partial U}{\partial R^{bi}}. \quad (5)$$

We derive a closed form for M^g from (5), by assuming that the spacecraft is on a circular orbit so that the norm of x is constant, and the size of the spacecraft is much smaller than the size of the orbit.

As shown in Fig. 1, the coordinate of the spacecraft position in the LVLH frame is $r_0 e_3$, where $r_0\in\mathbb{R}$ is the radius of the circular orbit, and $e_3=[0,\,0,\,1]^T$. Therefore, the position of the spacecraft in the inertial frame is given by $x=r_0R^{li}e_3$. Using this expression,

$$\begin{aligned}\frac{\partial U}{\partial R^{bi}}&=\int_{\mathcal{B}}\frac{GM\,r_0R^{li}\,e_3\rho^T}{\|\,r_0e_3+R^{bl}\rho\,\|^3}\,dm,\\&=\frac{GM}{r_0}\int_{\mathcal{B}}\frac{\left(R^{li}e_3\hat{\rho}^T\right)\frac{\|\rho\|}{r_0}}{\left[1+2\left(e_3^TR^{bl}\hat{\rho}\right)\frac{\|\rho\|}{r_0}+\frac{\|\rho\|^2}{r_0^2}\right]^{\frac{3}{2}}}\,dm,\end{aligned}$$

where $\hat{\rho}=\frac{\rho}{\|\rho\|}\in\mathbb{R}^3$ is the unit vector along the direction of ρ. Assume that the size of the spacecraft is significantly smaller than the size of the orbit, i.e. $\frac{\|\rho\|}{r_0}\ll 1$. Using a Taylor series expansion, we obtain the 2nd order approximation.

$$\frac{\partial U}{\partial R^{bi}}=\frac{GM}{r_0}\int_{\mathcal{B}}R^{li}e_3\hat{\rho}^T\left\{\frac{\|\rho\|}{r_0}-3e_3^TR^{bl}\hat{\rho}\frac{\|\rho\|^2}{r_0^2}\right\}\,dm.$$

Since the body fixed frame is located at the mass center of the spacecraft, $\int_{\mathcal{B}}\rho\;dm=0$. Therefore, the first term in the above equation vanishes. Because $e_3^TR^{bl}\rho$ is a scalar quantity, we can rewrite the above equation as

$$\frac{\partial U}{\partial R^{bi}}=-3\omega_0^2R^{li}e_3e_3^TR^{bl}\left(\frac{1}{2}\mathrm{tr}[J]\,I_{3\times3}-J\right), \quad (6)$$

where $\omega_0=\sqrt{\frac{GM}{r_0^3}}\in\mathbb{R}$ is the orbital angular velocity, and $J\in\mathbb{R}^{3\times3}$ is the moment of inertia matrix of the spacecraft. Substituting (6) into (5), and using the property $S(x\times y)=yx^T-xy^T$ for $x,y\in\mathbb{R}^3$, we obtain an expression for the gravity gradient moment as follows.

$$M^g=3\omega_0^2R^{bl^T}e_3\times JR^{bl^T}e_3. \quad (7)$$

Discrete equations of motion: In the continuous equations of motion, the structure of (2), (3), and (4) ensures that R^{bi}, R^{li}, and R^{bl} evolve on the special orthogonal group, SO(3). However, general numerical integration methods, including the popular Runge-Kutta methods, do not preserve the orthogonality property of this group. For example, if we integrate (4) by a typical Runge-Kutta scheme, the quantity $R^{bl^T}R^{bl}$ inevitably drifts from the identity matrix as the simulation time increases.

The rotation matrix is commonly parameterized by Euler angles or quaternions. These attitude kinematics equations can be numerically integrated and are used to recompute the rotation matrix. However, Euler angles are not global expressions of the attitude since they have associated singularities. The analytical expressions for sensitivities are hard to develop since many trigonometric terms are encountered. Quaternions have no singularity, but quaternions must lie on the three sphere $\mathbb{S}^3$. General numerical integration methods do not preserve the unit length of a quaternion. Therefore, quaternions have the same numerical drift problem as rotation matrices. Furthermore, quaternions, which are diffeomorphic to SU(2), double covers SO(3). So there are inevitable ambiguities in expressing the attitude.

These cause significant inaccuracies in numerical simulations based on quaternions and Euler angles. In particular, the gravity gradient moment, as given in (7), depends on R^{bl} directly, and consequently, errors in computing R^{bl} cause errors in the gravity gradient moment. These effects are more pronounced when the simulation time is large.

Lie group variational integrators preserve the orthonormal structure of SO(3) without any reprojection or parametrization. They also conserve the momentum map, and the symplectic property of rigid body dynamics. In addition, the total energy is well-behaved, as it only oscillates in a bounded fashion about its true value. So, Lie group variational integrators are geometrically exact. Using the results given in [3], a Lie group variational integrator for the attitude dynamics of a spacecraft in a circular orbit are given by

$$\Pi_{k+1} = F_k^T \Pi_k + \frac{h}{2} F_k^T M_k^g + \frac{h}{2} M_{k+1}^g, \tag{8}$$

$$hS(\Pi_k + \frac{h}{2} M_k^g) = F_k J_d - J_d F_k^T, \tag{9}$$

$$R_{k+1}^{bi} = R_k^{bi} F_k, \tag{10}$$

$$R_{k+1}^{li} = R_k^{li} e^{S(\omega_0 e_2)h}, \tag{11}$$

$$R_{k+1}^{bl} = e^{-S(\omega_0 e_2)h} R_k^{bl} F_k, \tag{12}$$

where the subscript k denotes variables at the kth time step, and $h \in \mathbb{R}$ is the integration step size. The matrix $J_d \in \mathbb{R}^{3\times 3}$ is a nonstandard moment of inertia matrix defined by $J_d = \frac{1}{2}\mathrm{tr}[J]\, I_{3\times 3} - J$.

The matrix $F_k = R_k^{bi^T} R_{k+1}^{bi} \in \mathrm{SO}(3)$ is the relative attitude between integration steps, and it is obtained by solving (9). Since F_k and $e^{S(\omega_0 e_2)h}$ are in $\mathrm{SO}(3)$, and $\mathrm{SO}(3)$ is closed under matrix multiplication, R_k^{bi}, R_k^{li}, and R_k^{bl} evolve in $\mathrm{SO}(3)$ automatically for all k according to (10), (11), and (12). The actual computation of F_k is done in the Lie algebra $\mathfrak{so}(3)$ of dimension 3, and the rotation matrices are updated by multiplication with the exponential of a skew-symmetric matrix. So, Lie group variational integrators are numerically efficient, and there is no excessive computational burden in updating the 9 elements of the rotation matrix. The properties of these discrete equations of motion are discussed more explicitly in [3] and [4].

Since ω_0 is constant, the solution of (11) is given by

$$R_k^{li} = R_0^{li} e^{S(\omega_0 e_2)kh}. \tag{13}$$

III. SPACECRAFT ATTITUDE MANEUVERS

A. Problem formulation

A two point boundary value problem is formulated for spacecraft rest-to-rest maneuver between two given attitudes for a fixed maneuver time. This is a Lambert type boundary value problem on $\mathrm{SO}(3)$. The initial attitude and the desired terminal attitude are expressed as rotation matrices with respect to the LVLH frame, namely R_0^{bl}, $R_{N_d}^{bl} \in \mathrm{SO}(3)$. Two impulsive control torques are applied at the initial time and the terminal time. We assume that the control torques are purely impulsive, which means that each impulse changes the angular velocity of the spacecraft instantaneously, but it does not have any effect on the attitude of the spacecraft at that instant. The rotational motion of the spacecraft between the initial time and the terminal time is uncontrolled. This assumption is reasonable when the operating time of the spacecraft control moment actuator is much smaller than the fixed maneuver time.

We define this boundary value problem directly on $\mathrm{SO}(3)$ instead of using parameterizations such as Euler angles and quaternions. This approach allows us to define the attitude of the spacecraft globally without singularities and ambiguities. We use the discrete equations of motion given in (8)–(12) for the problem formulation and for the following analysis.

We transform this two point boundary value problem into a nonlinear root finding problem. For a given initial angular momentum Π_0 and an initial attitude of the spacecraft R_0^{bl}, the terminal angular momentum Π_N and the terminal attitude R_N^{bl} are determined by the discrete equations of motion. By choosing the initial angular momentum so that the terminal attitude of the spacecraft is equal to the desired attitude, i.e. $R_N^{bl} = R_{N_d}^{bl}$, we obtain the initial impulse and the terminal impulse. Thus, the nonlinear boundary value problem for the spacecraft attitude maneuver is formulated as

$$\begin{gathered} \text{given} : R_0^{bl}, R_{N_d}^{bl}, N \\ \text{find} : \Pi_0 \\ \text{such that } R_N^{bl} = R_{N_d}^{bl} \text{ subject to (8)–(12)}, \end{gathered}$$

where $N \in \mathbb{N}$ is the number of integration steps determined by $N = \frac{T}{h}$ for the fixed maneuver time T and the fixed integration step size h.

B. Computational approach

We solve a sequence of linear boundary value problems whose solutions converge to the solution of the nonlinear boundary value problem.

Linearization: The equations of motion are linearized about a given trajectory, and they are expressed in terms of the Lie algebra $\mathfrak{so}(3)$. Consider small perturbations from a given trajectory denoted by Π_k^ϵ, $R_k^{bi,\epsilon}$, $R_k^{bl,\epsilon}$, F_k^ϵ:

$$\Pi_k^\epsilon = \Pi_k + \epsilon\delta\Pi_k, \tag{14}$$

$$R_k^{bi,\epsilon} = R_k^{bi} + \epsilon\delta R_k^{bi} + \mathcal{O}(\epsilon^2), \tag{15}$$

$$R_k^{bl,\epsilon} = R_k^{bl} + \epsilon\delta R_k^{bl} + \mathcal{O}(\epsilon^2), \tag{16}$$

$$F_k^\epsilon = F_k + \epsilon\delta F_k + \mathcal{O}(\epsilon^2), \tag{17}$$

where $\epsilon \in \mathbb{R}$. Since the orbital angular velocity is constant, $\delta R_k^{li} = 0$.

The infinitesimal variation of the angular momentum $\delta\Pi_k$ can be expressed in $\mathbb{R}^3$. The variation of the rotation matrix $R_k^{bi,\epsilon} \in \mathrm{SO}(3)$ can be expressed as

$$R_k^{bi,\epsilon} = R_k^{bi} e^{\epsilon S(\zeta_k)},$$

where $\zeta_k \in \mathbb{R}^3$ and $S(\zeta_k) \in \mathfrak{so}(3)$ is a skew-symmetric matrix. Since the map $S(\cdot)$ is an isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$, ζ_k is well defined. Then, the infinitesimal variation δR_k^{bi} is given by

$$\delta R_k^{bi} = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} R_k^{bi,\epsilon} = R_k^{bi} S(\zeta_k). \tag{18}$$

δF_k is obtained from definition (10), and (18) as

$$\begin{aligned} \delta F_k &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} R_k^{bi,\epsilon^T} R_{k+1}^{bi,\epsilon}, \\ &= -S(\zeta_k) F_k + F_k S(\zeta_{k+1}). \end{aligned} \tag{19}$$

Since $\delta R_k^{li} = 0$, δR_k^{bl} is given by

$$\delta R_k^{bl} = R_k^{li^T} \delta R_k^{bi} = R_k^{bl} S(\zeta_k). \tag{20}$$

In summary, equations (18), (19) and (20) describe the variations of rotation matrices in $\mathrm{SO}(3)$.

Substituting (14), (16), and (17) into (8), and (9), and ignoring higher-order terms, the linearized discrete equations of motion are given by

$$\delta\Pi_{k+1} = \delta F_k^T \Pi_k + F_k^T \delta\Pi_k + \frac{h}{2}\delta F_k^T M_k^g + \frac{h}{2} F_k^T \delta M_k^g + \frac{h}{2}\delta M_{k+1}^g, \tag{21}$$

$$hS(\delta\Pi_k + \frac{h}{2}\delta M_k^g) = \delta F_k J_d - J_d \delta F_k^T, \tag{22}$$

where

$$\delta M_k^g = 3\omega_0^2 \left[\delta R_k^{bl^T} e_3 \times J R_k^{bl^T} e_3 + R_k^{bl^T} e_3 \times J \delta R_k^{bl^T} e_3\right]. \tag{23}$$

These equations are not in standard form since (22) is an implicit equation in δF_k. By using (18), (19) and (20), we will obtain an explicit solution of (22), and rewrite the above equations in standard form.

Substituting (20) into (23), and using the property $S(x)y = -S(y)x$ for all $x, y \in \mathbb{R}^3$, δM_k^g can be written as

$$\begin{aligned}\delta M_k^g &= 3\omega_0^2 \Big[- S(J R_k^{bl^T} e_3) S(R_k^{bl^T} e_3) \\ &\qquad + S(R_k^{bl^T} e_3) J S(R_k^{bl^T} e_3)\Big]\zeta_k, \\ &= \mathcal{M}_k \zeta_k, \end{aligned} \tag{24}$$

where $\mathcal{M}_k \in \mathbb{R}^{3\times 3}$.

Using (19) and the property $S(Rx) = RS(x)R^T$ for $x \in \mathbb{R}^3$, $R \in \mathrm{SO}(3)$, the right hand side of (22) is written as

$$\begin{aligned}&\delta F_k J_d - J_d \delta F_k^T \\ &= -\left\{S(\zeta_k) F_k J_d + J_d F_k^T S(\zeta_k)\right\} \\ &\quad + \left\{S(F_k \zeta_{k+1}) F_k J_d + J_d F_k^T S(F_k \zeta_{k+1})\right\}. \end{aligned} \tag{25}$$

Substituting (24) and (25) into (22), and using the property $S(x)A + A^T S(x) = S(\{\mathrm{tr}[A]\, I_{3\times 3} - A\}x)$ for $x \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 3}$, (22) can be transformed into an equivalent vector form;

$$\begin{aligned} h\delta\Pi_k + \frac{h^2}{2}\mathcal{M}_k \zeta_k = &- \{\mathrm{tr}[F_k J_d]\, I_{3\times 3} - F_k J_d\}\,\zeta_k \\ &+ \{\mathrm{tr}[F_k J_d]\, I_{3\times 3} - F_k J_d\}\, F_k \zeta_{k+1}. \end{aligned}$$

Multiplying both sides by $F_k^T \{\mathrm{tr}[F_k J_d]\, I_{3\times 3} - F_k J_d\}^{-1}$ and rearranging, we obtain

$$\begin{aligned}\zeta_{k+1} &= F_k^T \left[\frac{h^2}{2} \{\mathrm{tr}[F_k J_d]\, I_{3\times 3} - F_k J_d\}^{-1} \mathcal{M}_k + I_{3\times 3}\right]\zeta_k \\ &\quad + h F_k^T \{\mathrm{tr}[F_k J_d]\, I_{3\times 3} - F_k J_d\}^{-1} \delta\Pi_k, \\ &\triangleq \mathcal{A}_k \zeta_k + \mathcal{B}_k \delta\Pi_k, \end{aligned} \tag{26}$$

where $\mathcal{A}_k, \mathcal{B}_k \in \mathbb{R}^{3\times 3}$. Substituting (26) into (19), δF_k is obtained as

$$\begin{aligned}\delta F_k &= -\eta_k F_k + F_k \eta_{k+1}, \\ &= -S(\zeta_k) F_k + F_k S(\zeta_{k+1}), \\ &= -S(\zeta_k) F_k + F_k S(\mathcal{A}_k \zeta_k + \mathcal{B}_k \delta\Pi_k). \end{aligned} \tag{27}$$

Equation (27) is an explicit solution of (22).

Now we rewrite (21) in the standard form of linear discrete equations of motion. Substituting (24), (27) into (21), and rearranging, we obtain

$$\begin{aligned}\delta\Pi_{k+1} &= \Bigg[F_k^T S(\Pi_k + \frac{h}{2} M_k^g)\,\{-I_{3\times 3} + F_k \mathcal{A}_k\} \\ &\qquad + \frac{h}{2} F_k^T \mathcal{M}_k + \frac{h}{2}\mathcal{M}_{k+1}\mathcal{A}_k\Bigg]\zeta_k \\ &\quad + \left[S(F_k^T \left\{\Pi_k + \frac{h}{2} M_k^g\right\})\mathcal{B}_k + F_k^T + \frac{h}{2}\mathcal{M}_{k+1}\mathcal{B}_k\right]\delta\Pi_k, \\ &\triangleq \mathcal{C}_k \zeta_k + \mathcal{D}_k \delta\Pi_k, \end{aligned} \tag{28}$$

where $\mathcal{C}_k, \mathcal{D}_k \in \mathbb{R}^{3\times 3}$.

In summary, (26) and (28) are linear discrete equations equivalent to (21), (22) and (23), and they can be written as

$$\begin{aligned}\begin{bmatrix}\zeta_{k+1} \\ \delta\Pi_{k+1}\end{bmatrix} &= \begin{bmatrix}\mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k\end{bmatrix}\begin{bmatrix}\zeta_k \\ \delta\Pi_k\end{bmatrix}, \\ &\triangleq A_k \begin{bmatrix}\zeta_k \\ \delta\Pi_k\end{bmatrix}, \end{aligned} \tag{29}$$

where $A_k \in \mathbb{R}^{6\times 6}$. Equation (29) is the linear discrete equation for perturbations of the attitude dynamics of a spacecraft in a circular orbit, expressed in terms of $\mathbb{R}^3 \simeq \mathfrak{so}(3)$. The important feature of (29) is that it is linearized in such a way that it respects the geometry of the special orthogonal group $\mathrm{SO}(3)$.

Linear boundary value problem: The solution of (29) is given by

$$\begin{aligned}\begin{bmatrix}\zeta_N \\ \delta\Pi_N\end{bmatrix} &= \left(\prod_{k=0}^{N-1} A_k\right)\begin{bmatrix}\zeta_0 \\ \delta\Pi_0\end{bmatrix}, \\ &\triangleq \begin{bmatrix}\Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22}\end{bmatrix}\begin{bmatrix}\zeta_0 \\ \delta\Pi_0\end{bmatrix}, \end{aligned} \tag{30}$$

where $\Phi_{ij} \in \mathbb{R}^{3\times 3}$ for $i, j = 1, 2$.

For the given boundary value problem, $\zeta_0 = 0$ since the initial attitude of the spacecraft is given and fixed, and $\delta\Pi_N$ is free since the terminal angular momentum is compensated by the terminal impulse. Then, we obtain

$$\zeta_N = \Phi_{12}\delta\Pi_0.$$

This equation provides Φ_{12}, the sensitivity derivative of the terminal attitude with respect to a change in the initial angular momentum. It states that for a given trajectory, if we update the initial angular velocity by $\delta\Pi_0$, then the terminal attitude is changed from R_N^{bl} to $R_N^{bl} e^{S(\zeta_N)} = R_N^{bl} e^{S(\Phi_{12}\delta\Pi_0)}$.

We choose the change of the initial angular momentum so that the updated terminal attitude is equal to the desired terminal attitude; $R_N^{bl} e^{S(\Phi_{12}\delta\Pi_0)} = R_{N_d}^{bl}$, or equivalently,

$$\delta\Pi_0 = \Phi_{12}^{-1} S^{-1}\Big(\mathrm{logm}\Big(R_N^{bl^T} R_{N_d}^{bl}\Big)\Big), \tag{31}$$

where $S^{-1}(\cdot) : \mathfrak{so}(3) \mapsto \mathbb{R}^3$ is the inverse mapping of $S(\cdot)$, and logm denotes the matrix logarithm. Equation (31) provides a solution of the linear boundary value problem for the attitude dynamics of a spacecraft in a circular orbit, assuming that Φ_{12} is invertible.

Nonlinear boundary value problem: The linear boundary value problem is solved successively so that its solution converges to the solution of the nonlinear boundary value problem. A numerical algorithm is summarized as follows.

Set Error $= 2\epsilon_S$.
Guess an initial condition $\Pi_0^{(0)}$.
Set $i = 0$.
while Error $> \epsilon_S$.
Find $\Pi_k^{(i)}, R_k^{bl(i)}$ using $\Pi_0^{(i)}$ and (8), (9), (12).
Compute the error;
$\zeta_N^{(i)} = S^{-1}\Big(\mathrm{logm}\Big({R_N^{bl(i)}}^T R_{N_d}^{bl}\Big)\Big)$, Error $= \left\|\zeta_N^{(i)}\right\|$.
Update the initial condition; $\Pi_0^{(i+1)} = \Pi_0^{(i)} + c\Phi_{12}^{-1}\zeta_N^{(i)}$.
Set $i = i + 1$.
end while

Here the superscript (i) denotes the ith iteration, and $\epsilon_S, c \in \mathbb{R}$ are a stopping criterion and a scaling factor, respectively.

This computational approach utilizes an exact and efficient method to compute sensitivity derivatives in the special orthogonal group $\mathrm{SO}(3)$. The sensitivity derivatives are then used to solve the two point boundary value problem.

C. Numerical example

Three spacecraft rest-to-rest maneuvers between relative equilibrium attitudes are considered. The resulting motions are highly nonlinear, large angle maneuvers.

The mass, length and time dimensions are normalized. The moment of inertia of the spacecraft and simulation parameters are chosen as $J = \mathrm{diag}\,[1,\ 2.8,\ 2]$, $\epsilon_s = 10^{-14}$, $c = 0.1$, $h = 0.001$. Each maneuver is completed in a quarter of the orbit, $T = \frac{\pi}{2}$. The boundary conditions and the corresponding computed impulsive control are as follows.

(i) Rotational maneuver about the LVLH axis e_1:

$$R_0^{bl} = I_{3\times 3}, \quad R_{N_d}^{bl} = \mathrm{diag}\,[1, -1, -1]\,,$$
$$\Pi_0 = [2.116,\ 1.531,\ -1.782]^T\,,$$
$$-\Pi_N = [-2.116,\ 1.531,\ 1.782]^T\,.$$

(ii) Rotational maneuver about the LVLH axes e_1 and e_2:

$$R_0^{bl} = \mathrm{diag}\,[1, -1, -1]\,, \quad R_{N_d}^{bl} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$
$$\Pi_0 = [-1.323,\ 1.798,\ 0.932]^T\,,$$
$$-\Pi_N = [0.397,\ -1.586,\ -1.310]^T\,.$$

(iii) Rotational maneuver about the LVLH axes e_2 and e_3:

$$R_0^{bl} = \mathrm{diag}\,[1, -1, -1]\,, \quad R_{N_d}^{bl} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\Pi_0 = [1.047,\ 0.437,\ 2.800]^T\,,$$
$$-\Pi_N = [-1.416,\ -1.761,\ 1.159]^T\,.$$

Fig. 2 shows the attitude maneuver of the spacecraft, and the angular velocity response for each case. (Simple animations which show these spacecraft maneuvers can be found at `http://www.umich.edu/~tylee`.)

IV. Optimal spacecraft attitude maneuvers

A. Problem formulation

An optimization problem is formulated as a rest-to-rest maneuver of an axially-symmetric spacecraft from a given initial attitude to a given terminal reduced attitude for a fixed maneuver time. The initial attitude is expressed by a rotation matrix with respect to the LVLH frame, namely R_0^{bl}. The terminal desired attitude is given by the reduced attitude, $\Lambda_{N_d} = R_N^{bl} e_3 \in \mathbb{S}^2$. This reduced attitude represents the direction of the spacecraft axis of symmetry e_3 in the LVLH frame. Two impulsive control moments are applied at the initial time and the terminal time, and the maneuver of the spacecraft between the initial time and the terminal time is uncontrolled.

In the problem studied in section III, the initial parameter Π_0 is exactly prescribed by the constraint $R_{N_d}^{bl} = R_N^{bl}$, and the discrete dynamics. In this section, we relax the terminal constraint by only specifying it up to a rotation about the axis of symmetry of the spacecraft. Then, we can formulate an optimal attitude maneuver problem.

The performance index is the sum of the magnitudes of the initial impulse and the terminal impulse. Equivalently, one can minimize the change in the initial angular momentum and the change in the terminal angular momentum. Since the initial attitude and the terminal time are fixed, Π_N and Λ_N can be considered as functions of Π_0 through the discrete equations of motion. The optimization problem is equivalent to

$$\begin{aligned} &\text{given : } R_0^{bl}, \Lambda_{N_d}, N, \\ &\min_{\Pi_0} \mathcal{J} = \left\|\Pi_0 - \omega_0 R_0^T J e_2\right\| + \left\|\omega_0 R_N^T J e_2 - \Pi_N\right\|, \\ &\quad\;\; = \|H_0\| + \|H_N\|\,, \\ &\text{such that } \mathcal{C} = \left\|\Lambda_N - \Lambda_{N_d}\right\|^2 = 0, \\ &\text{subject to (8)–(12).} \end{aligned}$$

B. Computational approach

This problem is optimized by the Sequential Quadratic Programming (SQP) method using analytical expressions for the sensitivity derivatives of the performance index and of the constraint equation.

The variation of the performance index is

$$\delta\mathcal{J} = \frac{H_0^T}{\|H_0\|}\delta\Pi_0 + \frac{H_N^T}{\|H_N\|}\left\{-\omega_0 S(\zeta_N) R_N^T J e_2 - \delta\Pi_N\right\}.$$

Since the initial attitude is given and fixed, the perturbation of the initial attitude ζ_0 is zero. Therefore $\delta\Pi_N = \Phi_{22}\delta\Pi_0$ and $\zeta_N = \Phi_{12}\delta\Pi_0$ from (30). Then, $\delta\mathcal{J}$ is given by

$$\delta\mathcal{J} = \left[\frac{H_0^T}{\|H_0\|} + \frac{H_N^T}{\|H_N\|}\left\{\omega_0 S(R_N^T J e_2)\Phi_{12} - \Phi_{22}\right\}\right]\delta\Pi_0. \tag{32}$$

Since Λ_{N_d} is fixed and $\Lambda_N \in \mathbb{S}^2$, the variation of the constraint can be written as

$$\delta\mathcal{C} = -2\Lambda_{N_d}^T \delta\Lambda_N = 2\Lambda_{N_d}^T R_N^{bl} S(e_3)\zeta_N,$$

where $\zeta_N = \Phi_{12}\delta\Pi_0$ from (30). Thus, $\delta\mathcal{C}$ is

$$\delta\mathcal{C} = \left[2\Lambda_d^T R_N^{bl} S(e_3)\Phi_{12}\right]\delta\Pi_0. \tag{33}$$

Equations (32) and (33) are analytical expressions for the sensitivity derivatives of the performance index and the constraint.

C. Numerical example

The mass, length and time dimensions are normalized. The moment of inertia of the spacecraft is chosen as $J = \mathrm{diag}\,[3,\, 3,\, 2]$, so that e_3 is the axis of symmetry of the spacecraft.

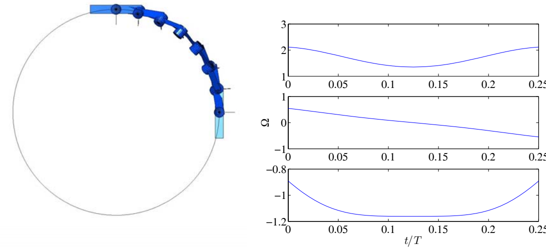
The desired maneuver is to rotate the axis of symmetry from the radial direction to the normal to the orbital plane during a quarter orbit. The boundary conditions are given by

$$R_0^{bl} = \mathrm{diag}\,[1,\, -1,\, -1]\,,\ \Lambda_{N_d} = [0,\, -1,\, 0]^T\,.$$

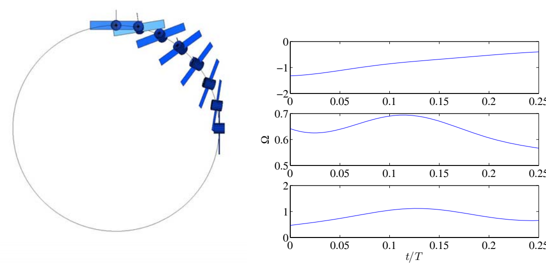
We use MATLAB's `fmincon` function as an optimization tool. The sensitivity derivatives of the performance index and the constraint are provided by (32) and (33). The initial guess of the initial angular momentum is chosen as $\Pi_0^{(0)} = J\,[1,\, 1,\, 0]^T$. The optimized performance index and the violation of constraints are $\mathcal{J} = 6.771$, $\mathcal{C} = 4.80 \times 10^{-14}$. The corresponding angular momenta and the terminal attitude are

$$\Pi_0 = [-2.915,\, -2.347,\, -2.734]^T\,,$$
$$-\Pi_N = [-0.343,\, 2.686,\, 2.734]^T\,,$$
$$R_N^{bl} = \begin{bmatrix} 0.633 & 0.733 & 0.000 \\ 0.000 & 0.000 & -1.000 \\ -0.773 & 0.633 & 0.0000 \end{bmatrix},$$

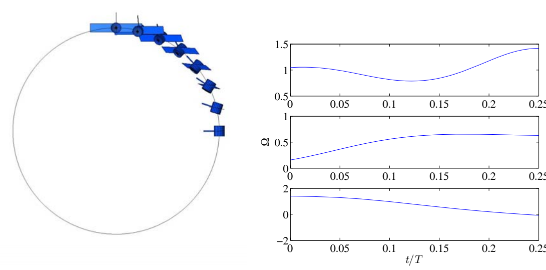
so that $\Lambda_N = R_N^{bl} e_3 = [0,\, 0,\, -1]^T = \Lambda_{N_d}$. Fig. 3 shows the optimal maneuver of the spacecraft.



(a) Rotation about the LVLH axis e_1



(b) Rotation about the LVLH axes e_1 and e_2



(c) Rotation about the LVLH axes e_2 and e_3

Fig. 2. Spacecraft attitude maneuvers

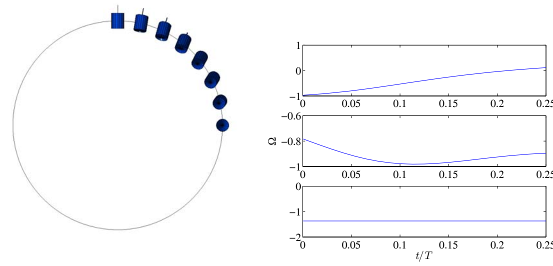


Fig. 3. Optimal spacecraft attitude maneuver

V. Conclusion

A global model for a rigid spacecraft in a circular orbit about a large central body is presented. This model includes gravity gradient effects that arise from the non-uniform gravity field.

The sensitivity derivatives for attitude dynamics of a rigid body are derived while satisfying the global geometry of the problem. Accurate computational approaches for solving a nonlinear boundary value problem and the minimal impulse optimal control problem for spacecraft attitude maneuvers are studied using sensitivity derivatives.

The attitude dynamics are represented by a rotation matrix in the Lie group $\mathrm{SO}(3)$, and it is updated by Lie group variational integrators that preserve the structure of $\mathrm{SO}(3)$ as well as other geometric invariants of motion. The sensitivity derivatives are expressed in terms of the Lie algebra $\mathfrak{so}(3)$. This approach completely avoids the singularities and ambiguities associated with Euler angles or quaternions, and it leads to a geometrically exact and numerically efficient method for rigid body attitude dynamics problems.

Although the development in this paper includes a gravity gradient moment and the rotation of the LVLH frame, the results presented reduce to the case of a free rigid body if $\omega_0 = 0$. That is, the computational approach suggested applies directly to attitude maneuvers of the free rigid body.

References

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