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Discrete Hamilton–Jacobi Theory and Discrete Optimal Control

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***Abstract*— We develop a discrete analogue of Hamilton–Jacobi theory in the framework of discrete Hamiltonian mechanics. The resulting discrete Hamilton–Jacobi equation is discrete only in time. The correspondence between discrete and continuous Hamiltonian mechanics naturally gives rise to a discrete analogue of Jacobi's solution to the Hamilton–Jacobi equation. We prove discrete analogues of Jacobi's solution to the Hamilton–Jacobi equation and of the geometric Hamilton–Jacobi theorem. These results are readily applied to the discrete optimal control setting, and some well-known results in discrete optimal control theory, such as the Bellman equation, follow immediately. We also apply the theory to discrete linear Hamiltonian systems, and show that the discrete Riccati equation follows as a special case.**

I. Introduction

A. Discrete Mechanics

Discrete mechanics is a reformulation of Lagrangian and Hamiltonian mechanics with discrete time, as opposed to a discretization of the equations in the continuous-time theory. It not only provides a systematic view of structure-preserving integrators, but also has interesting theoretical aspects analogous to continuous-time Lagrangian and Hamiltonian mechanics [see, e.g., 15; 17; 18]. The main feature of discrete mechanics is its use of discrete versions of variational principles. Namely, discrete mechanics assumes that the dynamics is defined at discrete times from the outset, formulates a discrete variational principle for such dynamics, and then derives a discrete analogue of the Euler–Lagrange or Hamilton's equations from it.

The advantage of this construction is that it naturally gives rise to discrete analogues of the concepts and ideas in continuous time that have the same or similar properties, such as symplectic forms, the Legendre transformation, momentum maps, and Noether's theorem [15]. Whereas the main topic in discrete mechanics is the development of structure-preserving algorithms for Lagrangian and Hamiltonian systems [see, e.g., 15], the theoretical aspects of it are interesting in their own right, and furthermore provide insight into the numerical aspects as well.

Another notable feature of discrete mechanics is that it is a generalization of (nonsingular) discrete optimal control problems. In fact, as stated in Marsden and West [15], discrete mechanics is inspired by discrete formulations of optimal control problems (see, e.g., Jordan and Polak [10] and Cadzow [5]).

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B. Hamilton–Jacobi Theory

In classical mechanics [see, e.g., 3; 8; 13; 14], the Hamilton–Jacobi equation is first introduced as a partial differential equation satisfied by the action integral. Specifically, let Q be a configuration space and T^*Q be its cotangent bundle, and suppose that $(\hat{q}(s), \hat{p}(s)) \in T^*Q$ is a solution of Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q},$$

where $H : T^*Q \to \mathbb{R}$ is the Hamiltonian of the system. Then calculate the action integral along the solution starting from $s = 0$ and ending at $s = t$ with $t > 0$:

$$S(q,t) := \int_0^t \Big[\hat{p}(s) \cdot \dot{\hat{q}}(s) - H(\hat{q}(s), \hat{p}(s))\Big]\, ds, \qquad (1)$$

where $q := \hat{q}(t)$ and we regard the resulting integral as a function of the endpoint $(q,t) \in Q \times \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of positive real numbers. Then by taking a variation of the endpoint (q,t), one obtains a partial differential equation satisfied by $S(q,t)$:

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0. \qquad (2)$$

This is the *Hamilton–Jacobi (H–J) equation*.

Conversely, it is shown that if $S(q,t)$ is a solution of the H–J equation then $S(q,t)$ is a generating function for the family of canonical transformations (or symplectic flow) that describe the dynamics defined by Hamilton's equations. This result is the theoretical basis for the powerful technique of exact integration called separation of variables.

The idea of H–J theory is also useful in optimal control theory [see, e.g., 11]. Namely, the Hamilton–Jacobi equation turns into the Hamilton–Jacobi–Bellman (HJB) equation, which is a partial differential equation satisfied by the optimal cost function. It is also shown that the costate of the optimal solution is related to the solution of the HJB equation.

C. Discrete Hamilton–Jacobi Theory

The main objective of this paper is to present a discrete analogue of H–J theory within the framework of discrete Hamiltonian mechanics [12].

There are some previous works on discrete-time analogues of the H–J equation, such as Elnatanov and Schiff [6] and Lall and West [12]. Specifically, Elnatanov and Schiff [6] derived an equation for a generating function of a coordinate transformation that trivializes the dynamics. This derivation is a discrete analogue of the conventional derivation of the continuous-time H–J equation [see, e.g., 13, Chapter VIII]. Lall and West [12] formulated a discrete Lagrangian analogue of the H–J equation as a separable optimization problem.

D. Main Results

Our work was inspired by the result of Elnatanov and Schiff [6], and further extends the result by developing discrete analogues of results in (continuous-time) H–J theory. Namely, we formulate a discrete analogue of Jacobi's solution, which relates the discrete action sum (see Eq. (3) below) with a solution of the discrete H–J equation. Another important result in this paper is a discrete analogue of the H–J theorem, which relates the solution of the discrete H–J equation with the solution of the discrete Hamilton's equations.

We also show that the discrete H–J equation is a generalization of the discrete Riccati equation and the Bellman equation (discrete HJB equation). (See Fig. 1.) Specifically, we establish a link with discrete-time optimal control theory, and show that the Bellman equation of dynamic programming follows. This link makes it possible to interpret discrete analogues of Jacobi's solution and the H–J theorem in the optimal control setting. Namely, we show that these results reduce to two well-known results in optimal control theory that relate the Bellman equation with the optimal solution. We also show that the discrete H–J equation applied to linear discrete Hamiltonian systems reduces to the discrete Riccati equation. This is again a discrete analogue of the well-known result that the H–J equation applied to linear Hamiltonian systems reduces to the Riccati equation [see, e.g., 11, p. 421].

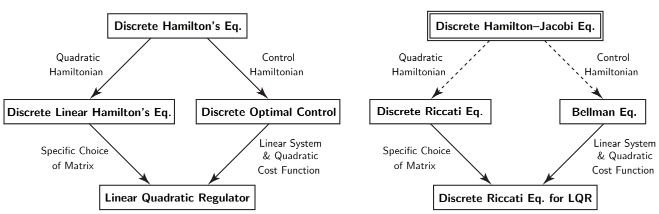

Fig. 1. Discrete evolution equations (left) and corresponding discrete H–J-type equations (right). Dashed lines are the links established in the paper.

E. Outline

We first present a brief review of discrete Lagrangian and Hamiltonian mechanics in Section II. In Section III we describe a discrete analogue of Jacobi's solution to the discrete H–J equation, and also discuss the left and right variants and more explicit forms of the discrete H–J equation. In Section IV we prove a discrete version of the H–J theorem. Section V establishes the link with discrete-time optimal control and interprets the results of the preceding sections in this setting. In Section VI we apply the theory to linear discrete Hamiltonian systems, and show that the discrete Riccati equation follows from the discrete H–J equation.

II. Discrete Mechanics

This section briefly reviews some key results of discrete mechanics following Marsden and West [15] and Lall and West [12].

A. Discrete Lagrangian Mechanics

A discrete Lagrangian flow $\{q_k\}$ for $k = 0, 1, \ldots, N$ on an n-dimensional differentiable manifold Q can be described based on the following discrete variational principle. Let $S_{\rm d}^N$ be the following action sum of the discrete Lagrangian $L_{\rm d} : Q \times Q \to \mathbb{R}$:

$$S_{\rm d}^N(\{q_k\}_{k=0}^N) := \sum_{k=0}^{N-1} L_{\rm d}(q_k, q_{k+1}) \approx \int_0^{t_N} L(q(t), \dot{q}(t))\, dt, \tag{3}$$

where $L : TQ \to \mathbb{R}$ is the Lagrangian of the corresponding continuous system.

Consider discrete variations $q_k \mapsto q_k + \epsilon\delta q_k$ for $k = 0, 1, \ldots, N$ with $\delta q_0 = \delta q_N = 0$. Then the discrete variational principle $\delta S_{\rm d}^N = 0$ gives the discrete Euler–Lagrange equations:

$$D_2 L_{\rm d}(q_{k-1}, q_k) + D_1 L_{\rm d}(q_k, q_{k+1}) = 0, \tag{4}$$

where D_i stands for the partial derivative with respect to the variable(s) in the i-th slot. This determines the discrete flow $F_{L_{\rm d}} : Q \times Q \to Q \times Q$:

$$F_{L_{\rm d}} : (q_{k-1}, q_k) \mapsto (q_k, q_{k+1}), \tag{5}$$

B. Discrete Hamiltonian Mechanics

Lall and West [12] introduced discrete Hamiltonian mechanics in the following way: Introduce the *right and left discrete Legendre transforms* $\mathbb{F}L_{\rm d}^{\pm} : Q \times Q \to T^*Q$ by

$$\mathbb{F}L_{\rm d}^{+} : (q_k, q_{k+1}) \mapsto (q_{k+1}, D_2 L_{\rm d}(q_k, q_{k+1})), \tag{6a}$$

$$\mathbb{F}L_{\rm d}^{-} : (q_k, q_{k+1}) \mapsto (q_k, -D_1 L_{\rm d}(q_k, q_{k+1})). \tag{6b}$$

With the discrete Legendre transform

$$p_{k+1} = \mathbb{F}L_{\rm d}^{+}(q_k, q_{k+1}) = D_2 L_{\rm d}(q_k, q_{k+1}), \tag{7}$$

we can define the following *right discrete Hamiltonian*:

$$H_{\rm d}^{+}(q_k, p_{k+1}) = p_{k+1} \cdot q_{k+1} - L_{\rm d}(q_k, q_{k+1}). \tag{8}$$

Then the discrete Hamiltonian map $\tilde{F}_{L_{\rm d}} : (q_k, p_k) \mapsto (q_{k+1}, p_{k+1})$ is defined implicitly by the *right discrete Hamilton's equations*

$$q_{k+1} = D_2 H_{\rm d}^{+}(q_k, p_{k+1}), \tag{9a}$$

$$p_k = D_1 H_{\rm d}^{+}(q_k, p_{k+1}). \tag{9b}$$

Similarly, with the discrete Legendre transform

$$p_k = \mathbb{F}L_{\rm d}^{-}(q_k, q_{k+1}) = -D_1 L_{\rm d}(q_k, q_{k+1}), \tag{10}$$

we can define the following *left discrete Hamiltonian*:

$$H_{\rm d}^{-}(p_k, q_{k+1}) = -p_k \cdot q_k - L_{\rm d}(q_k, q_{k+1}). \tag{11}$$

Then we have the *left discrete Hamilton's equations*

$$q_k = -D_1 H_{\rm d}^{-}(p_k, q_{k+1}), \tag{12a}$$

$$p_{k+1} = -D_2 H_{\rm d}^{-}(p_k, q_{k+1}). \tag{12b}$$

III. Discrete Hamilton–Jacobi Equation

A. Discrete Analogue of Jacobi's Solution

This section presents a discrete analogue of Jacobi's solution. This also gives an alternative derivation of the discrete H–J equation that is much simpler than that of Elnatanov and Schiff [6].

Theorem 1 *Consider the action sums Eq.* (3) *written in terms of the right discrete Hamiltonian, Eq.* (8)*:*

$$S_{\rm d}^k(q_k) := \sum_{l=0}^{k-1}\left[p_{l+1}\cdot q_{l+1} - H_{\rm d}^+(q_l, p_{l+1})\right] \qquad (13)$$

evaluated along a solution of the right discrete Hamilton's equations (9)*; each $S_{\rm d}^k(q_k)$ is seen as a function of the end point coordinates q_k and the discrete end time k. Then these action sums satisfy the* right discrete H–J equation

$$S_{\rm d}^{k+1}(q_{k+1}) - S_{\rm d}^k(q_k) - DS_{\rm d}^{k+1}(q_{k+1})\cdot q_{k+1} + H_{\rm d}^+\big(q_k, DS_{\rm d}^{k+1}(q_{k+1})\big) = 0, \qquad (14)$$

where $DS_{\rm d}^k(q_k) = (\partial S_{\rm d}^k/\partial q_k^1, \dots, \partial S_{\rm d}^k/\partial q_k^n)$.

Proof: From Eq. (13), we have

$$S_{\rm d}^{k+1}(q_{k+1}) - S_{\rm d}^k(q_k) = p_{k+1}\cdot q_{k+1} - H_{\rm d}^+(q_k, p_{k+1}), \quad (15)$$

where p_{k+1} is considered to be a function of q_k and q_{k+1}, i.e., $p_{k+1} = p_{k+1}(q_k, q_{k+1})$. Taking the derivative of both sides with respect to q_{k+1} and using Eq. (12a), we have

$$p_{k+1} = DS_{\rm d}^{k+1}(q_{k+1}). \qquad (16)$$

Substituting this into Eq. (15) gives Eq. (14). ∎

Remark 2 *Recall that, in the derivation of the continuous H–J equation [see, e.g., 7, Section 23], we consider the variation of the action integral Eq.* (1) *with respect to the end point (q, t) and find*

$$dS = p\,dq - H(q,p)\,dt. \qquad (17)$$

This gives

$$\frac{\partial S}{\partial t} = -H(q,p), \qquad p = \frac{\partial S}{\partial q}, \qquad (18)$$

and hence the H–J equation (2)*. Table I summarizes the correspondence between the ingredients in the continuous and discrete theories (see also Remark 2).*

B. The Right and Left Discrete H–J Equations

We can also write the action sum Eq. (3) in terms of the left discrete Hamiltonian, Eq. (11), as follows:

$$S_{\rm d}^k(q_k) = \sum_{l=0}^{k-1}\left[-p_l\cdot q_l - H_{\rm d}^-(p_l, q_{l+1})\right]. \qquad (19)$$

Then we can proceed as in the proof of Theorem 1 (see Ohsawa et al. [16] for details) to obtain the *left discrete H–J equation*:

$$S_{\rm d}^{k+1}(q_{k+1}) - S_{\rm d}^k(q_k) + DS_{\rm d}^k(q_k)\cdot q_k + H_{\rm d}^-\big(DS_{\rm d}^k(q_k), q_{k+1}\big) = 0. \quad (20)$$

TABLE I

Correspondence between ingredients in continuous and discrete theories; $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers and $\mathbb{N}_0$ is the set of non-negative integers.

Continuous	**Discrete**
$(q,t)\in Q\times\mathbb{R}_{\geq 0}$	$(q_k,k)\in Q\times\mathbb{N}_0$
$\dot q = \partial H/\partial p$, $\dot p = -\partial H/\partial q$	$q_{k+1} = D_2 H_{\rm d}^+(q_k, p_{k+1})$, $p_k = D_1 H_{\rm d}^+(q_k, p_{k+1})$
$S(q,t) := \int_0^t p\,\dot q - H\,ds$	$S_{\rm d}^k(q_k) :=$ $\sum_{l=0}^{k-1}\left[p_{l+1}\cdot q_{l+1} - H_{\rm d}^+(q_l,p_{l+1})\right]$
$dS = \frac{\partial S}{\partial q}\,dq + \frac{\partial S}{\partial t}\,dt$	$S_{\rm d}^{k+1}(q_{k+1}) - S_{\rm d}^k(q_k)$
$p\,dq - H(q,p)\,dt$	$p_{k+1}\cdot q_{k+1} - H_{\rm d}^+(q_k,p_{k+1})$
$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0$	$S_{\rm d}^{k+1}(q_{k+1}) - S_{\rm d}^k(q_k)$ $- DS_{\rm d}^{k+1}(q_{k+1})\cdot q_{k+1}$ $+ H_{\rm d}^+\left(q_k, DS_{\rm d}^{k+1}(q_{k+1})\right) = 0$

As mentioned above, Eqs. (13) and (19) are the same action sum Eq.(3) expressed in different ways. Therefore we may summarize the above argument as follows:

Proposition 3 *The action sums, Eq.* (13) *or equivalently Eq.* (19)*, satisfy both the right and left discrete H–J equations* (14) *and* (20)*.*

C. Explicit Forms of the Discrete H–J Equations

The expressions for the right and left discrete H–J equations in Eqs. (14) and (20) are implicit in the sense that they contain two spatial variables q_k and q_{k+1}. However, Theorem 1 suggests that q_k and q_{k+1} may be considered to be related by the dynamics defined by either Eq. (9) or (12). More specifically, we may write q_{k+1} in terms of q_k. This results in explicit forms of the discrete H–J equations, and we shall *define* the discrete H–J equations by the resulting explicit forms.

For the right discrete H–J equation (14), we first define the map $f_k^+ : Q \to Q$ as follows: Replace p_{k+1} in Eq. (9a) by $DS_{\rm d}^{k+1}(q_{k+1})$ as suggested by Eq. (16):

$$q_{k+1} = D_2 H_{\rm d}^+\big(q_k, DS_{\rm d}^{k+1}(q_{k+1})\big). \qquad (21)$$

Assuming this equation is solvable for q_{k+1}, we define $f_k^+ : Q \to Q$ by the resulting q_{k+1}, i.e., f_k^+ is implicitly defined by

$$f_k^+(q_k) = D_2 H_{\rm d}^+\big(q_k, DS_{\rm d}^{k+1}(f_k^+(q_k))\big). \qquad (22)$$

We may now identify q_{k+1} with $f_k^+(q_k)$ in the implicit form of the right H–J equation (14):

$$S_{\rm d}^{k+1}(f_k^+(q)) - S_{\rm d}^k(q) - DS_{\rm d}^{k+1}(f_k^+(q))\cdot f_k^+(q) + H_{\rm d}^+\big(q, DS_{\rm d}^{k+1}(f_k^+(q))\big) = 0, \quad (23)$$

where we suppressed the subscript k of q_k since it is now clear that q_k is an independent variable as opposed to a function of the discrete time k. We *define* Eq. (23) to be the *right discrete H–J equation*. Notice that these are differential-difference equations defined on $Q \times \mathbb{N}$, with the spatial variable q and the discrete time k.

For the left discrete H–J equation (20), we define the map $f_k^- : Q \to Q$ as follows:

$$f_k^-(q_k) := \pi_Q \circ \tilde{F}_{L_\mathrm{d}}\left(dS_\mathrm{d}^k(q_k)\right), \tag{24}$$

where $\pi_Q : T^*Q \to Q$ is the cotangent bundle projection; equivalently, f_k^- is defined so that the diagram below commutes.

$$\begin{array}{ccc} T^*Q & \xrightarrow{\tilde{F}_{L_\mathrm{d}}} & T^*Q \\ {\scriptstyle dS_\mathrm{d}^k}\uparrow & & \downarrow{\scriptstyle \pi_Q} \\ Q & \underset{f_k^-}{\dashrightarrow} & Q \end{array} \qquad \begin{array}{ccc} dS_\mathrm{d}^k(q_k) & \longmapsto & \tilde{F}_{L_\mathrm{d}}\big(dS_\mathrm{d}^k(q_k)\big) \\ \uparrow & & \downarrow \\ q_k & \dashrightarrow & f_k^-(q_k) \end{array} \tag{25}$$

Notice also that, since the map $\tilde{F}_{L_\mathrm{d}} : (q_k, p_k) \mapsto (q_{k+1}, p_{k+1})$ is defined by Eq. (12), f_k^- is defined implicitly by

$$q_k = -D_1 H_\mathrm{d}^-\left(DS_\mathrm{d}^k(q_k), f_k^-(q_k)\right). \tag{26}$$

In other words, replace p_k in Eq. (12a) by $DS_\mathrm{d}^k(q_k)$, and define $f_k^-(q_k)$ as the q_{k+1} in the resulting equation.

We may now identify q_{k+1} with $f_k^-(q_k)$ in Eq. (20):

$$S_\mathrm{d}^{k+1}(f_k^-(q)) - S_\mathrm{d}^k(q) + DS_\mathrm{d}^k(q) \cdot q + H_\mathrm{d}^-\left(DS_\mathrm{d}^k(q), f_k^-(q)\right) = 0, \tag{27}$$

where we again suppressed the subscript k of q_k. We *define* Eqs. (23) and (27) to be the *right and left discrete H–J equations*, respectively. Notice that these are differential-difference equations defined on $Q \times \mathbb{N}$, with the spatial variable q and the discrete time k.

D. Discrete H–J Equation and Generating Functions

Assuming the uniqueness of the solution of the discrete H–J equation, Theorem 1 tells us that, the solution $S_\mathrm{d}^n(q)$ is the action integral written in terms of the end time and end point, which is the generating function of the the dynamics $(q_0, p_0) \mapsto (q_n, p_n)$. If we construct $\tilde{F}_{S_\mathrm{d}^n}$ from S_d^n using the correspondence[1] $L_\mathrm{d} \mapsto \tilde{F}_{L_\mathrm{d}}$, then $\tilde{F}_{S_\mathrm{d}^n}$ gives the map $(q_0, p_0) \mapsto (q_n, p_n)$, i.e., we have

$$\tilde{F}_{S_\mathrm{d}^n} = \underbrace{\tilde{F}_{L_\mathrm{d}} \circ \cdots \circ \tilde{F}_{L_\mathrm{d}}}_{n}, \tag{28}$$

In other words, a solution of the discrete H–J equation generates the n-step dynamics of the corresponding discrete Lagrangian/Hamiltonian system (see also Fig. 2).

Remark 4 *Note that Eq.* (28) *holds* exactly. *This property is not guaranteed for those solutions obtained by direct discretizations of the H–J equation.*

[1]Recall that a discrete Lagrangian is nothing but a generating function. See [15] and [12].

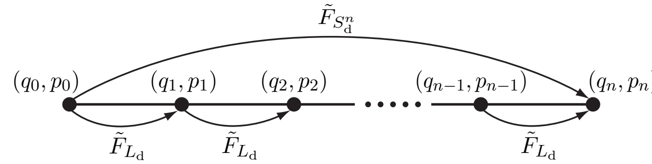

Fig. 2. The generating function S_d^n generates the flow defined by the n-fold composition $\tilde{F}_{L_\mathrm{d}} \circ \cdots \circ \tilde{F}_{L_\mathrm{d}}$.

IV. Discrete Hamilton–Jacobi Theorem

The following gives a discrete analogue of the geometric H–J theorem (Theorem 5.2.4) by Abraham and Marsden [1]:

Theorem 5 (Discrete Hamilton–Jacobi) *Suppose that S_d^k satisfies the right discrete H–J equation* (23)*, and let $\{c_k\}_{k=0}^N \subset Q$ be a set of points such that*

$$c_{k+1} = f_k^+(c_k) \quad \text{for} \quad k = 0, 1, \ldots, N-1. \tag{29}$$

Then the set of points $\{(c_k, p_k)\}_{k=0}^N \subset T^*Q$ with

$$p_k := DS_\mathrm{d}^k(c_k) \tag{30}$$

is a solution of the right discrete Hamilton's equations (9)*.*

Similarly, suppose that S_d^k satisfies the left discrete H–J equation (27)*, and let $\{c_k\}_{k=0}^N \subset Q$ be a set of points that satisfy*

$$c_{k+1} = f_k^-(c_k) \quad \text{for} \quad k = 0, 1, \ldots, N-1. \tag{31}$$

Furthermore, assume that the Jacobian Df_k^- is invertible at each point c_k. Then the set of points $\{(c_k, p_k)\}_{k=0}^N \subset T^*Q$ with

$$p_k := DS_\mathrm{d}^k(c_k) \tag{32}$$

is a solution of the left discrete Hamilton's equations (12)*.*

Proof: See Ohsawa et al. [16]. ■

V. Relation to the Discrete-Time HJB Equation

In this section we apply the above results to the optimal control setting. We will show that the (right) discrete H–J equation (23) gives the Bellman equation (discrete-time HJB equation) as a special case.

A. Discrete Optimal Control Problem

Let $\{q_k\}_{k=0}^N$ be the state variables in a vector space $V \cong \mathbb{R}^n$ with q_0 and q_N fixed and $u_\mathrm{d} := \{u_k\}_{k=0}^N$ be controls in the set $U \subset \mathbb{R}^m$. With a given function $C_\mathrm{d} : V \times U \to \mathbb{R}$, define the cost functional

$$J_\mathrm{d} := \sum_{k=0}^{N-1} C_\mathrm{d}(q_k, u_k). \tag{33}$$

Then a typical *discrete optimal control problem* is formulated as follows [see, e.g., 4; 5; 9; 10]:

Problem 6 *Minimize the cost functional, i.e.,*

$$\min_{u_\mathrm{d}} J_\mathrm{d} = \min_{u_\mathrm{d}} \sum_{k=0}^{N-1} C_\mathrm{d}(q_k, u_k) \tag{34}$$

subject to the constraint

$$q_{k+1} = f(q_k, u_k). \tag{35}$$

B. Necessary Condition for Optimality and the Discrete-Time HJB Equation

We would like to formulate the necessary condition for optimality. First introduce the augmented cost functional:

$$\begin{aligned}\hat{S}_{\mathrm{d}}^k(q_{\mathrm{d}}, p_{\mathrm{d}}, u_{\mathrm{d}}) &:= \sum_{l=0}^{k-1} \{C_{\mathrm{d}}(q_l, u_l) + p_{l+1} \cdot [q_{l+1} - f(q_l, u_l)]\} \\ &= \sum_{l=0}^{k-1} \Big[p_{l+1} \cdot q_{l+1} - \hat{H}_{\mathrm{d}}^+(q_l, p_{l+1}, u_l) \Big],\end{aligned}$$

where we defined the Hamiltonian

$$\hat{H}_{\mathrm{d}}^+(q_l, p_{l+1}, u_l) := p_{l+1} \cdot f(q_l, u_l) - C_{\mathrm{d}}(q_l, u_l) \tag{36}$$

and the shorthand notation $q_{\mathrm{d}} := \{q_l\}_{l=0}^k$, $p_{\mathrm{d}} := \{p_l\}_{l=1}^k$, and $u_{\mathrm{d}} := \{u_l\}_{l=0}^{k-1}$. Then the optimality condition Eq. (34) is restated as $\min_{q_{\mathrm{d}}, p_{\mathrm{d}}, u_{\mathrm{d}}} \hat{J}_{\mathrm{d}}^k(q_{\mathrm{d}}, p_{\mathrm{d}}, u_{\mathrm{d}})$ or equivalently

$$\min_{q_{\mathrm{d}}, p_{\mathrm{d}}, u_{\mathrm{d}}} \hat{S}_{\mathrm{d}}^k(q_{\mathrm{d}}, p_{\mathrm{d}}, u_{\mathrm{d}}). \tag{37}$$

In particular, extremality with respect to the control u_{d} implies

$$D_3 \hat{H}_{\mathrm{d}}^+(q_l, p_{l+1}, u_l) = 0, \quad l = 0, 1, \dots, k-1. \tag{38}$$

Now we assume that $\hat{H}_{\mathrm{d}}^+$ is sufficiently regular so that the optimal control $u_{\mathrm{d}}^* := \{u_l^*\}_{l=0}^{k-1}$ is determined by

$$D_3 \hat{H}_{\mathrm{d}}^+(q_l, p_{l+1}, u_l^*) = 0, \quad l = 0, 1, \dots, k-1. \tag{39}$$

Therefore u_l^* is a function of q_l and p_{l+1}, i.e., $u_l^* = u_l^*(q_l, p_{l+1})$. Then we can eliminate u_{d} in the minimization problem Eq. (37):

$$\min_{q_{\mathrm{d}}, p_{\mathrm{d}}} S_{\mathrm{d}}(q_{\mathrm{d}}, p_{\mathrm{d}}) = \min_{q_{\mathrm{d}}, p_{\mathrm{d}}} \sum_{l=0}^{k-1} \big[p_{l+1} \cdot q_{l+1} - H_{\mathrm{d}}^+(q_l, p_{l+1}) \big], \tag{40}$$

where we defined $H_{\mathrm{d}}^+(q_l, p_{l+1}) := \hat{H}_{\mathrm{d}}^+(q_l, p_{l+1}, u_l^*)$ and $S_{\mathrm{d}}^k(q_{\mathrm{d}}, p_{\mathrm{d}}) := \hat{S}_{\mathrm{d}}^k(q_{\mathrm{d}}, p_{\mathrm{d}}, u_{\mathrm{d}}^*)$. So now the problem is reduced to minimizing an action sum that has exactly the same form as the one in Eq. (13) formulated in the framework of discrete Hamiltonian mechanics.

The corresponding right discrete Hamilton's equations are

$$\begin{aligned} q_{k+1} &= f(q_k, u_k^*), \\ p_k &= p_{k+1} \cdot D_1 f(q_k, u_k^*) - D_1 C_{\mathrm{d}}(q_k, u_k^*). \end{aligned} \tag{41}$$

Therefore Eq. (22) gives the implicit definition of f_k^+ as follows:

$$f_k^+(q_k) = f\big(q_k, u_k^*\big(q_k, DS_{\mathrm{d}}^{k+1}(f_k^+(q_k))\big)\big). \tag{42}$$

Hence the (right) discrete H–J equation (23) applied to this case gives

$$S_{\mathrm{d}}^{k+1}(f(q_k, u_k^*)) - S_{\mathrm{d}}^k(q_k) - C_{\mathrm{d}}(q_k, u_k^*) = 0, \tag{43}$$

or equivalently

$$\min_{u_k} \big[S_{\mathrm{d}}^{k+1}(f(q_k, u_k)) - C_{\mathrm{d}}(q_k, u_k) \big] - S_{\mathrm{d}}^k(q_k) = 0, \tag{44}$$

which is the *Bellman equation* [see, e.g., 4].

C. Relation between the Discrete H–J and HJB Equations and its Consequences

Summarizing the observation made above, we have

Proposition 7 *The right discrete H–J equation* (23) *applied to the Hamiltonian formulation of the discrete optimal control problem 6 gives the Bellman equation* (44)*.*

Reinterpreting Theorems 1 and 5 in terms of this observation leads to the following well-known facts:

Proposition 8 *The optimal cost function satisfies the Bellman equation* (44)*.*

Proposition 9 *Let $S_{\mathrm{d}}^k(q_k)$ be a solution to the Bellman equation* (44)*. Then the costate p_k in the discrete maximum principle is given as follows:*

$$p_k = DS_{\mathrm{d}}^k(c_k), \tag{45}$$

where $c_{k+1} = f(c_k, u_k^*)$ with the optimal control u_k^*.

VI. Application To Discrete Linear Hamiltonian Systems

A. Discrete Linear Hamiltonian Systems and the Matrix Riccati Equation

Example 10 (Discrete linear Hamiltonian systems) *Consider a discrete Hamiltonian system on $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ (the configuration space is $Q = \mathbb{R}^n$) defined by the quadratic left discrete Hamiltonian*

$$H_{\mathrm{d}}^-(p_k, q_{k+1}) = \frac{1}{2} p_k^T M^{-1} p_k + p_k^T L q_{k+1} + \frac{1}{2} q_{k+1}^T K q_{k+1}, \tag{46}$$

where M, K, and L are real $n \times n$ matrices; we assume that M and L are invertible and also that M and K are symmetric. The left discrete Hamilton's equations (12) *become*

$$\begin{pmatrix} q_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} -L^{-1} & -L^{-1}M^{-1} \\ KL^{-1} & KL^{-1}M^{-1} - L^T \end{pmatrix} \begin{pmatrix} q_k \\ p_k \end{pmatrix}. \tag{47}$$

Now let us solve the left discrete H–J equation (27) *for this system. It is possible to show (see Ohsawa et al. [16]) that, for this particular case, the solution S_{d}^k takes the form*

$$S_{\mathrm{d}}^k(q) = \frac{1}{2} q^T A_k q + b_k^T q + c_k \tag{48}$$

where A_k are symmetric $n \times n$ matrices, b_k are elements in $\mathbb{R}^n$, and c_k are in $\mathbb{R}$. We substitute the above expression into the discrete H–J equation to find the equations for A_k, b_k, and c_k. Notice first that the map f_k^- is given by the first half of Eq. (47) *with p_k replaced by $DS_{\mathrm{d}}^k(q)$:*

$$f_k^-(q) = -L^{-1}(I + M^{-1}A_k)q - L^{-1}M^{-1}b_k. \tag{49}$$

Then substituting Eq. (48) *into the left-hand side of the left discrete H–J equation* (27) *yields the following recurrence* *relations for A_k, b_k, and c_k:*

$$A_{k+1} = L^T(I + A_k M^{-1})^{-1} A_k L - K, \tag{50a}$$

$$b_{k+1} = -L^T(I + A_k M^{-1})^{-1} b_k, \tag{50b}$$

$$c_{k+1} = c_k - \frac{1}{2} b_k^T (M + A_k)^{-1} b_k, \tag{50c}$$

where we assumed that $I + A_k M^{-1}$ is invertible.

Remark 11 *For the A_{k+1} defined by Eq.* (50a) *to be symmetric, it is sufficient that A_k is invertible; for if it is, then Eq.* (50a) *becomes*

$$A_{k+1} = L^T(A_k^{-1} + M^{-1})^{-1} L - K,$$

where A_k, M, and K are symmetric.

Remark 12 *We can rewrite Eq.* (50a) *as follows:*

$$A_{k+1} = \left[K L^{-1} + (K L^{-1} M^{-1} - L^T) A_k\right] \times (-L^{-1} - L^{-1} M^{-1} A_k)^{-1}. \tag{51}$$

Notice the exact correspondence between the coefficients in the above equation and the matrix entries in the discrete linear Hamiltonian equations (47)*. In fact, this is the discrete Riccati equation that corresponds to the iteration defined by Eq.* (47)*. See Ammar and Martin [2] for details on this correspondence.*

To summarize the above observation, we have

Proposition 13 *The discrete H–J equation* (27) *applied to the discrete linear Hamiltonian system* (47) *yields the discrete Riccati equation* (51)*.*

In other words, the discrete H–J equation is a nonlinear generalization of the discrete Riccati equation.

VII. Conclusion and Future Work

We developed a discrete-time analogue of the H–J theory starting from the discrete variational Hamilton equations formulated by Lall and West [12]. We showed that it possesses theoretical significance in discrete mechanics that is equivalent to that of the (continuous-time) H–J equation in Hamiltonian mechanics. Furthermore, we showed that the discrete H–J equation specializes to the Bellman equation if applied to discrete optimal control problems, and also that it reduces to the discrete Riccati equation with a quadratic Hamiltonian. This again gives discrete analogues of the corresponding known results in the continuous-time theory. Application to discrete optimal control also revealed that Theorems 1 and 5 specialize to two well-known results in discrete optimal control theory.

We are interested in the following topics for future work: (i) Application to integrable discrete systems; (ii) Development of numerical methods based on the discrete H–J equation; (iii) Extension to discrete nonholonomic and Dirac mechanics; (iv) Relation to the power method and iterations on the Grassmannian manifold.

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