

Constructing equivalence-preserving Dirac variational integrators with forces

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The dynamical motion of mechanical systems possesses underlying geometric structures and preserving these structures in numerical integration improves the qualitative accuracy and reduces the long-time error of the simulation. For a single mechanical system, structure preservation can be achieved by adopting the variational integrator construction (Marsden, J. & West, M. (2001) Discrete mechanics and variational integrators. *Acta Numer.*, **10**, 357–514). This construction has been generalized to more complex systems involving forces or constraints as well as to the setting of Dirac mechanics (Leok, M. & Ohsawa, T. (2011) Variational and geometric structures of discrete Dirac measures. *Found. Comput. Math.*, **11**, 529–562). Forced Lagrange–Dirac systems are described by a Lagrangian and an external force pair, and two pairs of Lagrangians and external forces are said to be equivalent if they yield the same equations of motion. However, the variational discretization of a forced Lagrange–Dirac system discretizes the Lagrangian and forces separately, and will generally depend on the choice of representation. In this paper we derive a class of Dirac variational integrators with forces that yield well-defined numerical methods that are independent of the choice of representation. We present a numerical simulation to demonstrate how such equivalence-preserving discretizations avoid spurious solutions that otherwise arise from poorly chosen representations.

Keywords: geometric numerical integration; variational integrators; forced Lagrangian mechanics.

1. Introduction

Reframing analytic mechanics as geometric mechanics by considering mechanical systems as evolving on symplectic or Poisson manifolds reveals the depth of geometric structure in the physical world. Examining the range of geometric structures and their associated invariants—momenta, energy, symplectic forms, etc.—relevant to a particular mechanical motion provides a qualitative picture of the motion. Computational geometric mechanics allows these geometric structures to guide the development of numerical methods, leading to improved qualitative behavior of the numerical solutions.

Analytical mechanics and its geometric properties can be derived from variational or almost-variational principles. Variational integrators mimic this process by deriving integrators as the discrete evolution equations implied by a discrete variational principle. This process was first developed for discretizing Hamilton’s principle in the case of conservative, nondegenerate, unconstrained Lagrangian systems. The discrete Hamilton’s principle produces a set of discrete evolution equations known as the discrete Euler–Lagrange (DEL) equations. Considered as a one-step map, the DEL equations are symplectic, approximately energy conserving and satisfy a discrete Noether’s theorem. This mirrors the symplecticity, energy conservation and momentum conservation at the continuous level.

The variational construction has been extended to accommodate forcing, holonomic constraints (Kane *et al.*, 2000; Marsden & West, 2001) and nonholonomic constraints (Cortés & Martinez, 2001;

McLachlan & Perlmutter, 2006). Most recently, the variational construction was applied to Lagrange–Dirac mechanical systems, which is the Lagrangian view of Dirac mechanics (Leok & Ohsawa, 2011). Dirac mechanics generalizes both Lagrangian and Hamiltonian mechanics to a formulation that accommodates forces and constraints as well as degeneracy. Forces, constraints and degeneracy each change the geometric structure of the system away from straightforward conservation laws, and Dirac geometry and mechanics elucidates the specifics of these changes (Yoshimura & Marsden, 2006a,b).

Dirac mechanics was also recently shown to be a useful setting for the study of interconnected systems (Jacobs & Yoshimura, 2014). Interconnected systems consist of multiple mechanical components joined together to operate as a whole. Many practical engineering systems can be conceptualized this way, and often it is more natural to model each component individually than to attempt to model the entire large-scale system monolithically. At the continuous level, interconnections can be represented using either constraints or interaction forces, which is related to the equivalence between the principle of virtual work and the d’Alembert principle for ideal constraints, i.e., those for which the net virtual work vanishes. But, at the discrete level, the relationship between these two approaches remains open. In Parks & Leok (2017), using the constraint approach to interconnection, we developed the foundations for incorporating the ease of subsystem-level modeling to geometric integration through an interconnected discrete Dirac simulation framework. That framework is based on incorporating the ideas of Jacobs & Yoshimura (2014) into the discrete foundation developed in Leok & Ohsawa (2011).

Towards a formulation of discrete interconnected systems using interaction forces, we extend Dirac variational integrators to a formulation that includes discrete forcing. Connected subsystems can be described by interaction forces, so accommodating forces may prove useful in interconnected applications. We can also view the interconnected modeling process as breaking a large (possibly conservative) system into smaller, more manageable parts. From this point of view, interaction forces are artificial modeling constructs, and we should ensure that they cancel properly in the final integration, so as to ensure that the large monolithic model is equivalent to the model based on interconnecting the component subsystems. This is analogous to the requirement that when representing a system as a collection of subsystems using free body diagrams the internal forces have to cancel. This paper addresses that concern by examining notions of equivalence at both the continuous and discrete levels and by providing a criterion for defining forced variational or Dirac integrators that are well defined with respect to changes in representation of equivalent systems.

The paper begins with a review of classic, forced and Dirac variational integrators in Section 2. Section 3 examines order of accuracy and equivalence preservation in forced variational integrators, including several numerical examples. This study informs the later design and implementation of forced Dirac variational integrators. Section 4 describes the extension of Dirac variational integrators to forced Dirac variational integrators, closing with a numerical example. Section 5 summarizes the implications of this work and suggests future research directions.

2. Background: a review of variational integrators

2.1 *Classic variational integrators*

Consider a conservative, unconstrained, nondegenerate Lagrangian system. Given the Lagrangian $L : TQ \rightarrow \mathbb{R}$ we can construct a discrete Lagrangian $L_d : Q \times Q \rightarrow \mathbb{R}$, which can be viewed as approximating the type I generating function for the symplectic time- h flow map of the Euler–Lagrange

vector field. With this, variational integrators are constructed from the discrete Hamilton's principle, which states that the discrete action sum is stationary,

$$\delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) = 0,$$

for fixed endpoints, $\delta q_0 = \delta q_N = 0$. Computing the infinitesimal variation of the discrete action sum and collecting the terms involving δq_k yield the DEL equations,

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) = 0.$$

Here $D_1 L_d$ and $D_2 L_d$ denote the derivative of the discrete Lagrangian L_d with respect its first and second arguments, respectively. We view $Q \times Q$ as the discrete analogue of the tangent bundle TQ and the DEL equations as defining a one-step map from $Q \times Q$ to itself that determines (q_k, q_{k+1}) from (q_{k-1}, q_k) .

One can also define a discrete flow along T^*Q by introducing two discrete Legendre transforms, $\mathbb{F}^\pm L_d : Q \times Q \rightarrow T^*Q$,

$$\mathbb{F}^+ L_d(q_k, q_{k+1}) = (q_{k+1}, D_2 L_d(q_k, q_{k+1})),$$

$$\mathbb{F}^- L_d(q_k, q_{k+1}) = (q_k, -D_1 L_d(q_k, q_{k+1})).$$

These give two possible definitions for the momentum at q_k , which are defined in terms of the data (q_k, q_{k+1}) and (q_{k-1}, q_k) ,

$$p_{k,k+1}^- = \mathbb{F}^- L_d(q_k, q_{k+1}) = -D_1 L_d(q_k, q_{k+1}),$$

$$p_{k-1,k}^+ = \mathbb{F}^+ L_d(q_{k-1}, q_k) = D_2 L_d(q_{k-1}, q_k).$$

Then the DEL equations enforce a momentum matching condition along the discrete flow, i.e.,

$$p_{k-1,k}^+ = p_{k,k+1}^-.$$

Thus, the momentum $p_k = p_{k-1,k}^+ = p_{k,k+1}^-$ is well defined along solutions of the DEL equations and we can define a one-step map in phase space by

$$p_{k+1} = D_2 L_d(q_k, q_{k+1}),$$

$$p_k = -D_1 L_d(q_k, q_{k+1}).$$

This map is called the discrete Hamiltonian map corresponding to the DEL equations.

The DEL equations and discrete Hamiltonian map each preserve the appropriate symplectic form and satisfy a discrete Noether's theorem, so that symmetries in the discrete Lagrangian L_d result in conservation of the component of the discrete momenta in the direction of the infinitesimal generators of the symmetry. As symplectic integrators they also approximately conserve energy over exponentially long time scales. These conservation properties hold and ensure that the qualitative behavior of the Lagrangian system is well approximated for any choice of L_d .

The choice of L_d is guided by the existence of an exact discrete Lagrangian, L_d^E , which yields a discrete Hamiltonian map that samples the exact Hamiltonian flow. For points $q_0, q_1 \in Q$ and h sufficiently small, let $q_{01}(t)$ be the unique curve such that $q_{01}(0) = q_0$, $q_{01}(h) = q_1$ and $q_{01}(t)$ satisfies the Euler–Lagrange equations for L on $[0, h]$. Then L_d^E is given by

$$L_d^E(q_0, q_1; h) = \int_0^h L(q_{01}(t), \dot{q}_{01}(t)) dt.$$

This is related to Jacobi’s solution of the Hamilton–Jacobi equation and can be viewed as evaluating the action integral on the solution of an Euler–Lagrange boundary-value problem, or it can equivalently be described variationally,

$$L_d^E(q_0, q_1; h) = \underset{\substack{q \in C^2([0, h], Q) \\ q(0) = q_0, q(h) = q_1}}{\text{ext}} \int_0^h L(q(t), \dot{q}(t)) dt.$$

One can approximate the exact discrete Lagrangian L_d^E by incorporating a wide variety of standard numerical techniques in order to obtain a computable discrete Lagrangian L_d (Leok & Shingel, 2012; Hall & Leok, 2015). By the variational error analysis described in Marsden & West (2001), it can be shown that if the discrete Lagrangian $L_d(q_0, q_1; h) = L_d^E(q_0, q_1; h) + \mathcal{O}(h^{r+1})$, then the associated discrete Hamiltonian map is an order r approximation of the exact flow map.

2.2 Forced variational integrators

Given a continuous force $f_L : TQ \rightarrow T^*Q$ that is a fiber-preserving map over the identity, we introduce discrete forces $f_d^\pm : Q \times Q \rightarrow T^*Q$ that approximate the virtual work of the forces. Then the forced DEL equations are derived from the discrete Lagrange–d’Alembert principle,

$$\delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) + \sum_{k=0}^{N-1} [f_d^-(q_k, q_{k+1}) \cdot \delta q_k + f_d^+(q_k, q_{k+1}) \cdot \delta q_{k+1}] = 0, \quad (2.1)$$

where as before, the endpoints are fixed, $\delta q_0 = \delta q_N = 0$. The forced DEL equations are given by

$$D_2 L_d(q_{k-1}, q_k) + f_d^+(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + f_d^-(q_k, q_{k+1}) = 0.$$

The discrete Legendre transforms and discrete Hamiltonian map now incorporate the contribution of the discrete forces,

$$p_{k+1} = \mathbb{F}^{f^+} L_d(q_k, q_{k+1}) = D_2 L_d(q_k, q_{k+1}) + f_d^+(q_k, q_{k+1}), \quad (2.2a)$$

$$p_k = \mathbb{F}^{f^-} L_d(q_k, q_{k+1}) = -D_1 L_d(q_k, q_{k+1}) - f_d^-(q_k, q_{k+1}). \quad (2.2b)$$

With these momentum definitions, the forced DEL equations can again be viewed as a momentum-matching condition.

These equations reduce to the DEL equations in the absence of forcing and they satisfy a forced discrete Noether's theorem for symmetries of the discrete Lagrangian where the discrete forces do no work in the direction of the infinitesimal generators of the symmetry. Forced variational integrators exhibit better energy behavior than nongeometric integrators in practice, in the sense that the discrete energy evolution better reflects the exact energy evolution of the system. This lacks rigorous explanation since the forced equations are no longer symplectic nor energy preserving at either the continuous or the discrete level.

Once again, exact discrete quantities guide both the error analysis and the practical choice of L_d and f_d^\pm . For points $q_0, q_1 \in Q$, define $q_{01}(t)$ as above, except that we now require $q_{01}(t)$ to satisfy the *forced* Euler–Lagrange equations,

$$\frac{\partial L}{\partial q}(q, \dot{q}) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right) + f_L(q, \dot{q}) = 0,$$

for L and f_L on $[0, h]$. Then L_d^E and $f_d^{E\pm}$ are given by

$$\begin{aligned} L_d^E(q_0, q_1, h) &= \int_0^h L(q_{01}(t), \dot{q}_{01}(t)) \, dt, \\ f_d^{E+}(q_0, q_1, h) &= \int_0^h f_L(q_{01}(t), \dot{q}_{01}(t)) \cdot \frac{\partial q_{01}(t)}{\partial q_1} \, dt, \\ f_d^{E-}(q_0, q_1, h) &= \int_0^h f_L(q_{01}(t), \dot{q}_{01}(t)) \cdot \frac{\partial q_{01}(t)}{\partial q_0} \, dt. \end{aligned}$$

We obtain computable L_d and f_d^\pm by approximating the curve $q_{01}(t)$ that satisfies the forced Euler–Lagrange boundary-value problem and the integrals that arise in the definition of $L_d^E, f_d^{E\pm}$. Again, the orders of these approximations determine the order of accuracy of the discrete Hamiltonian map. This fact is stated without proof in [Marsden & West \(2001\)](#) and we provide an explicit proof in the following section.

2.3 Dirac variational integrators

Dirac variational integrators were developed in [Leok & Ohsawa \(2011\)](#) and they generalize the variational integrator construction to the case of Dirac mechanics. In [Leok & Ohsawa \(2011\)](#), the authors develop both the variational theory of discrete Dirac mechanics and explicit discrete analogues of Dirac structures. The authors arrive at two formulations, (+)- and (−)-discrete Dirac mechanics, stemming from the choice of generating function used to define the discrete analogue of the symplectic flat map, $\Omega_{d\pm}^b$. Discretization begins from a continuous problem described by a Lagrangian function $L : TQ \rightarrow \mathbb{R}$ and a continuous constraint distribution $\Delta_Q \subset TQ$. The constraint distribution is determined by its annihilator, which we write as $\Delta_Q^\circ = \text{span}\{\omega^a\}_{a=1}^m \subset T^*Q$. In each formulation of discrete Dirac mechanics, we construct a discrete Lagrangian L_d and discrete annihilating one-forms $\omega_{d\pm}^a$ based on the continuous problem and a choice of retraction $\mathcal{R} : TQ \rightarrow Q$.

2.3.1 (+)-discrete Dirac mechanics. In (+)-discrete Dirac mechanics we define the discrete one-forms as

$$\omega_{d+}^a(q_k, q_{k+1}) = \omega^a(q_k, \mathcal{R}_{q_k}^{-1}(q_{k+1})).$$

These then define a discrete distribution as:

$$\Delta_Q^{d+} = \{(q_k, q_{k+1}) \in Q \times Q \mid \omega_{d+}^a(q_k, q_{k+1}) = 0, a = 1, \dots, m\}.$$

The (+)-discrete Lagrange–d’Alembert–Pontryagin principle is

$$\delta \sum_{k=0}^{N-1} [L_d(q_k, q_k^+) + p_{k+1}(q_{k+1} - q_k^+)] = 0,$$

with variations that vanish at the endpoints, $\delta q_0 = \delta q_N = 0$, and discrete constraints, $(q_k, q_{k+1}) \in \Delta_Q^{d+}$. We also impose a constraint on the variations $\delta q_k \in \Delta_Q(q_k)$ after computing variations inside the sum. The variable q_k^+ serves as the discrete analogue to the introduction of v in the continuous principle. This process produces the (+)-discrete Lagrange–Dirac equations of motion,

$$\begin{aligned} 0 &= \omega_{d+}^a(q_k, q_{k+1}), \\ q_{k+1} &= q_k^+ \\ p_{k+1} &= D_2 L_d(q_k, q_k^+), \\ p_k &= -D_1 L_d(q_k, q_k^+) + \mu_a \omega^a(q_k), \end{aligned}$$

where $a = 1, \dots, m$, μ_a are Lagrange multipliers that impose the condition that $p_k + D_1 L_d(q_k, q_k^+) \in \Delta_Q^\circ(q_k)$ and the last equation uses the Einstein summation convention. These equations simplify to the DEL equations in the unconstrained case and they recover the nonholonomic integrators of [Cortés & Martínez \(2001\)](#). They can also be expressed in terms of discrete Dirac structures as

$$(X_d^k, \mathfrak{D}^+ L_d(q_k, q_k^+)) \in D_{\Delta_Q}^{d+}, \quad (2.3)$$

where $X_d^k = ((q_k, p_k), (q_{k+1}, p_{k+1}))$ is the discrete vector field, $\mathfrak{D}^+ L_d$ is the (+)-discrete Dirac differential and $D_{\Delta_Q}^{d+}$ is the (+)-discrete Dirac structure induced by the continuous distribution Δ_Q . The discrete symplectic flat map, Ω_{d+}^b , contributes to the definition of $D_{\Delta_Q}^{d+}$. We show (2.3) now to highlight its similarity to the continuous expression for constrained Dirac mechanics, $(X, \mathfrak{D}L) \in D_{\Delta_Q}$. We provide explicit descriptions of the discrete objects as needed in the development of forced Dirac integrators below.

2.3.2 (−)-discrete Dirac mechanics. This formulation defines the discrete one-forms as

$$\omega_{d-}^a(q_k, q_{k+1}) = \omega^a(q_{k+1}, \mathcal{R}_{q_{k+1}}^{-1}(q_k))$$

and uses the $(-)$ -discrete Lagrange–d’Alembert–Pontryagin principle

$$\delta \sum_{k=0}^{N-1} [L_d(q_{k+1}^-, q_{k+1}) - p_k(q_k - q_{k+1}^-)] = 0.$$

We impose the same constraints on the variations of the discrete principle as in the $(+)$ case, using Δ_Q^{d-} defined from $\{\omega_{d-}^a\}_{a=1}^m$ instead of Δ_Q^{d+} . The variable q_k^- now plays the role of the discrete velocity. This yields the $(-)$ -discrete Lagrange–Dirac equations

$$\begin{aligned} 0 &= \omega_{d-}^a(q_k, q_{k+1}), \\ q_k &= q_{k+1}^-, \\ p_k &= -D_1 L_d(q_{k+1}^-, q_{k+1}), \\ p_{k+1} &= D_2 L_d(q_{k+1}^-, q_{k+1}) + \mu_a \omega^a(q_{k+1}). \end{aligned}$$

Equivalently,

$$(X_d^k, \mathfrak{D}^- L_d(q_{k+1}^-, q_{k+1})) \in D_{\Delta_Q^{d-}}^{d-}.$$

Again, we avoid defining each discrete object until necessary and the interested reader is referred to [Leok & Ohsawa \(2011\)](#) for the details.

3. Forced integrators revisited

In [Marsden & West \(2001\)](#), the authors assert without proof the existence of a theorem relating the order of a forced variational integrator to the order of approximation of L_d, f_d^\pm to $L_d^E, f_d^{E\pm}$. Here we provide an explicit proof of the most practical direction of that theorem and a recipe for constructing forced integrators of known order. We then analyse the conditions under which this process yields a well-defined, equivalence-preserving integrator and discuss the implications of equivalence preservation in interconnected applications. The section closes with several numerical examples illustrating both our construction process and the consequences of equivalence preservation.

3.1 Determining the order of a forced variational integrator

Theorem 3.1 below explicitly proves that the order of approximation of L_d and f_d^\pm to L_d^E and $f_d^{E\pm}$ determine the order of accuracy of the discrete Hamiltonian map they define. This gives the most useful direction of the order theorem mentioned in [Marsden & West \(2001\)](#), since the order of L_d and f_d^\pm are easier to calculate than the order of the integrator.

We need a few preliminary definitions before we state the theorem. First, we explicitly define what we mean by the order of L_d, f_d^\pm and $\mathbb{F}^{f^\pm} L_d$, using the same definitions as in [Marsden & West \(2001\)](#). Thus, a given L_d is of order r if there exist constants $C_L > 0, h_L > 0$ and an open subset $U_L \subset TQ$ with compact closure such that

$$\|L_d(q(0), q(h), h) - L_d^E(q(0), q(h), h)\| \leq C_L h^{r+1}$$

for all solutions $q(t)$ of the forced Euler–Lagrange equations with initial conditions satisfying $(q(0), \dot{q}(0)) \in U_L$ and for all $h \leq h_L$. Similarly, f_d^\pm are of order r if there exist constants $C_{f_\pm} > 0$, $h_{f_\pm} > 0$ and open subsets $U_{f_\pm} \subset TQ$ with compact closure such that

$$\|f_d^\pm(q(0), q(h), h) - f_d^{E\pm}(q(0), q(h), h)\| \leq C_{f_\pm} h_{f_\pm}^{r+1}$$

for all solutions $q(t)$ of the forced Euler–Lagrange equations with initial conditions satisfying $(q(0), \dot{q}(0)) \in U_f^\pm$ and for all $h \leq h_f^\pm$. We will say that L_d, f_d^\pm are *simultaneously of order r* if there exists $U \subset U_L \cap U_f^+ \cap U_f^-$ such that U is a nontrivial open set with compact closure. For the discrete Legendre transforms to be of order r we require constants $C^\pm > 0$, $h^\pm > 0$ and an open set $U^\pm \subset TQ$ such that

$$\|\mathbb{F}^{f^\pm} L_d(q(0), q(h), h) - \mathbb{F}^{f^\pm} L_d^E(q(0), q(h), h)\| \leq C^\pm (h^\pm)^{r+1}$$

for all solutions $q(t)$ of the forced Euler–Lagrange equations with initial conditions satisfying $(q(0), \dot{q}(0)) \in U^\pm$ and for all $h \leq h^\pm$.

THEOREM 3.1 Consider a hyperregular Lagrangian L with force f_L , corresponding Hamiltonian H and corresponding Hamiltonian force f_H . If L_d and f_d^\pm are simultaneously of order r , then

- a. the forced discrete Legendre transforms $\mathbb{F}^{f^\pm} L_d$ are of order r ;
- b. the discrete Hamiltonian map is of order r .

Proof.

- a. Assume that L_d and f_d^\pm are simultaneously of order r . Then L_d of order r implies existence of a function e such that

$$L_d(q(0), q(h), h) = L_d^E(q(0), q(h), h) + h^{r+1} e(q(0), q(h), h) \quad (3.1)$$

and $\|e(q(0), q(h), h)\| \leq C_L$ on U . Similarly, f_d^+ of order r implies the existence of a function e^+ such that

$$f_d^+(q(0), q(h), h) = f_d^{E+}(q(0), q(h), h) + h^{r+1} e^+(q(0), q(h), h) \quad (3.2)$$

and $\|e^+(q(0), q(h), h)\| \leq C_{f^+}$ on U .

Taking derivatives of (3.1) with respect to $q(h)$ gives

$$D_2 L_d(q(0), q(h), h) = D_2 L_d^E(q(0), q(h), h) + h^{r+1} D_2 e(q(0), q(h), h) .$$

We assume both L_d and L_d^E have continuous derivatives in $q(h)$, so, by the definition of e in (3.1) $D_2 e$ is continuous and bounded on the compact set $\text{cl}(U)$. Combining this with (3.2) we have

$$\begin{aligned} & D_2 L_d(q(0), q(h), h) + f_d^+(q(0), q(h), h) \\ &= D_2 L_d^E(q(0), q(h), h) + f_d^{E+}(q(0), q(h), h) \\ &+ h^{r+1} [D_2 e(q(0), q(h), h) + e^+(q(0), q(h), h)] . \end{aligned}$$

From the above arguments, $D_2e(q(0), q(h), h) + e^+(q(0), q(h), h)$ is bounded on the set $\text{cl}(U)$. Thus, $\mathbb{F}^{f^+}L_d$ is of order r . Taking derivatives of L_d with respect to $q(0)$ and using a similar calculation with f_d^- shows that $\mathbb{F}^{f^-}L_d$ is of order r .

- b. Let $F_{L_d}^f$ denote the integrator defined by the forced DEL equations and $\tilde{F}_{L_d}^f$ denote the discrete Hamiltonian integrator defined by the forced discrete Legendre transforms in (2.2). From the definitions of $F_{L_d}^f, \tilde{F}_{L_d}^f$ and $\mathbb{F}^{f^\pm}L_d$ we have the commutative diagram (3.3), which is the forced equivalent of diagram (1.5.3) in Marsden & West (2001).

$$\begin{array}{ccccc}
 & (q_0, q_1) & \xrightarrow{F_{L_d}^f} & (q_1, q_2) & \\
 & \swarrow \mathbb{F}^{f^-}L_d & & \swarrow \mathbb{F}^{f^-}L_d & \\
 & (q_0, p_0) & \xrightarrow{\tilde{F}_{L_d}^f} & (q_1, p_1) & \xrightarrow{\tilde{F}_{L_d}^f} & (q_2, p_2) \\
 & \searrow \mathbb{F}^{f^+}L_d & & \searrow \mathbb{F}^{f^+}L_d &
 \end{array} \tag{3.3}$$

Thus, we can express the forced discrete Hamiltonian map as $\tilde{F}_{L_d}^f = \mathbb{F}^{f^+}L_d \circ (\mathbb{F}^{f^-}L_d)^{-1}$, directly analogous to the unforced discrete Hamiltonian map, $\tilde{F}_{L_d} = \mathbb{F}^+L_d \circ (\mathbb{F}^-L_d)^{-1}$. As such, the proof of (b) from (a) given in Marsden & West (2001) carries over exactly and we choose not to reproduce it here. \square

The next theorem provides a recipe for constructing forced triples (L_d, f_d^\pm) of known order.

THEOREM 3.2 Consider a p th-order accurate boundary-value solution method and a q th-order accurate quadrature formula. Assume that both the Lagrangian and the forcing function are Lipschitz continuous in both variables. Use the boundary-value solution method to obtain approximations, $(q^i, v^i) \approx (q(c_i h), v(c_i h))$, at the quadrature nodes c_i of the solution $(q(t), v(t))$ of the forced Euler-Lagrange boundary-value problem. Then the associated discrete Lagrangian given by

$$L_d(q_0, q_1; h) = h \sum_{i=0}^n b_i L(q^i, v^i)$$

and the discrete forces given by

$$\begin{aligned}
 f_d^+(q_0, q_1; h) &= h \sum_{i=0}^n b_i f(q^i, v^i) \frac{\partial q^i}{\partial q_1}, \\
 f_d^-(q_0, q_1; h) &= h \sum_{i=0}^n b_i f(q^i, v^i) \frac{\partial q^i}{\partial q_0},
 \end{aligned}$$

all have order of accuracy $\min(p + 1, q)$.

Proof. The result follows from a few straightforward calculations. We begin with L_d :

$$\begin{aligned}
L_d^E(q_0, q_1, h) &= \int_0^h L(q_{01}(t), \dot{q}_{01}(t)) \, dt \\
&= \left[h \sum_{i=1}^m b_i L(q_{01}(c_i h), \dot{q}_{01}(c_i h)) \right] + \mathcal{O}(h^{q+1}) \\
&= \left[h \sum_{i=1}^m b_i L(q^i + \mathcal{O}(h^{p+1}), v^i + \mathcal{O}(h^{p+1})) \right] + \mathcal{O}(h^{q+1}) \\
&= h \sum_{i=1}^m b_i L(q^i, v^i) + h \sum_{i=1}^m b_i \mathcal{O}(h^{p+1}) + \mathcal{O}(h^{q+1}) \\
&= L_d(q_0, q_1, h) + \mathcal{O}(h^{p+2}) + \mathcal{O}(h^{q+1}).
\end{aligned}$$

For f_d^+ ,

$$\begin{aligned}
f_d^{E+}(q_0, q_1, h) &= \int_0^h f(q_{01}(t), \dot{q}_{01}(t)) \cdot \frac{\partial q_{01}(t)}{\partial q_1} \, dt \\
&= \left[h \sum_{i=1}^m b_i f(q_{01}(c_i h), v_{01}(c_i h)) \cdot \frac{\partial q_{01}(c_i h)}{\partial q_1} \right] + \mathcal{O}(h^{q+1}) \\
&= \left[h \sum_{i=1}^m b_i f(q^i + \mathcal{O}(h^{p+1}), v^i + \mathcal{O}(h^{p+1})) \cdot \left(\frac{\partial q^i}{\partial q_1} + \mathcal{O}(h^{p+1}) \right) \right] + \mathcal{O}(h^{q+1}) \\
&= \left[h \sum_{i=1}^m b_i \left(f(q^i, v^i) + \mathcal{O}(h^{p+1}) \right) \cdot \left(\frac{\partial q^i}{\partial q_1} + \mathcal{O}(h^{p+1}) \right) \right] + \mathcal{O}(h^{q+1}) \\
&= h \sum_{i=1}^m b_i f(q^i, v^i) \cdot \frac{\partial q^i}{\partial q_1} + \mathcal{O}(h^{p+2}) + \mathcal{O}(h^{2p+3}) + \mathcal{O}(h^{q+1}) \\
&= f_d^+(q_0, q_1, h) + \mathcal{O}(h^{p+2}) + \mathcal{O}(h^{q+1}).
\end{aligned}$$

A similar calculation shows that

$$f_d^{E-}(q_0, q_1, h) = f_d^-(q_0, q_1, h) + \mathcal{O}(h^{p+2}) + \mathcal{O}(h^{q+1}).$$

□

Taken together, Theorems 3.1 and 3.2 show that we can construct forced variational integrators of known order from a choice of a quadrature rule and a boundary-value solution method. In fact, we could choose up to three different quadrature rules and boundary-value solution methods to define L_d , f_d^+ and f_d^- . The proofs above would still apply, and the resulting integrator would have order $\min(p_1 + 1, p_2 + 1, p_3 + 1, q_1, q_2, q_3)$ for p_i the orders of the boundary-value solutions and q_i the orders of the quadrature rules. However, this produces integrators with unpredictable results, as discussed in the next section.

3.2 Notions of equivalence and equivalence preservation

A forced Lagrangian system has equations of motion

$$\frac{\partial L}{\partial q}(q, \dot{q}) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right) + f_L(q, \dot{q}) = 0.$$

Since the equations of motion are defined by the combination of L and f_L , it is possible for pairs (L^1, f_L^1) and (L^2, f_L^2) with $L^1 \neq L^2$ and $f_L^1 \neq f_L^2$ to define the same equations of motion. We refer to this as *equivalence* of the pairs (L^1, f_L^1) and (L^2, f_L^2) . Most numerical methods simulate mechanics by numerically integrating the differential equations of motion, so they produce the same numerical approximation whether the motion was originally described using (L^1, f_L^1) or (L^2, f_L^2) .

Variational integrators simulate mechanics using the discrete equations of motion defined by a particular choice of L_d and f_d^\pm . Thus, a variational integrator is defined by the choice of discretizations $L \mapsto L_d$ and $f_L \mapsto f_d^\pm$. It is therefore natural to ask which discretization schemes for the discrete Lagrangian and discrete forces when applied to equivalent representations of forced Euler–Lagrange systems lead to equivalent discrete equations of motion. This reflects whether or not the resulting variational integrator is well defined on the equivalence class of representations of forced Euler–Lagrange systems.

More explicitly, given two equivalent representations (L^1, f_L^1) and (L^2, f_L^2) of a forced Euler–Lagrange system, the resulting solution trajectories are the same. Since a variational integrator for a forced Euler–Lagrange system aims to approximate that solution, it is desirable to consider well-defined variational integrators that produce the same approximation irrespective of which of the two equivalent representations (L^1, f_L^1) or (L^2, f_L^2) it is applied to. Well-definedness in this sense will be important in interconnected applications where we intentionally alter the forced representation of a system to view it as a collection of interacting subsystems. In particular, if we considered a system with many components in terms of the constituent free body diagrams, then it is essential when combining the free body diagrams for the internal forces to cancel out in order to recover the original system.

We discuss the implications of equivalence at the continuous and discrete levels. Then we provide a simple criterion for constructing well-defined forced variational integrators.

3.2.1 Continuous equivalence. Suppose we have two canonical Lagrangians, $L^i = \frac{1}{2} \dot{q}^T M^i \dot{q} - V^i(q)$. Then equality of the equations of motion implies

$$-\nabla V^1(q) + M^1 \ddot{q} + f_L^1(q, \dot{q}) = -\nabla V^2(q) + M^2 \ddot{q} + f_L^2(q, \dot{q}),$$

so that

$$f_L^1(q, \dot{q}) - f_L^2(q, \dot{q}) = \nabla(V^2 - V^1)(q) + (M^2 - M^1) \ddot{q}.$$

Comparing variables on each side we conclude that $M^2 = M^1$ and $f_L^1(q, v) - f_L^2(q, v) = \nabla V^1(q) - \nabla V^2(q)$.

More generally, assume two Lagrangians of the form $L^i(q, v) = K(q, v) - V^i(q) = \frac{1}{2} g(v, v) - V^i(q)$ for some metric tensor g . Then continuous equivalence again implies $f_L^1(q, v) - f_L^2(q, v) = \nabla V^2(q) - \nabla V^1(q)$.

3.2.2 *Notions of discrete equivalence.* In Marsden & West (2001), the authors define *strongly equivalent* discrete Lagrangians to be those that generate equivalent discrete Hamiltonian maps and *weakly equivalent* discrete Lagrangians to be those that generate equivalent discrete Lagrangian maps. That is to say that strongly equivalent discrete Lagrangians will generate the same discrete sequence $\{(q_k, p_k)\}_{k=0}^N \subset T^*Q$, whereas weakly equivalent discrete Lagrangians will generate only the same sequence $\{q_k\}_{k=0}^N \subset Q$.

We can define strong and weak equivalence of discrete triples $(L_d^i, f_d^{i\pm})$ in the same way. From diagram (3.3) we see that we have strong equivalence if and only if we have equivalence of the forced discrete Legendre transforms, $\mathbb{F}^{f\pm}L_d$, just as in the unforced case. Then Theorem 3.1 still holds for triples that are strongly equivalent to a triple meeting the assumptions. Moreover, Dirac variational integrators directly generalize the discrete Hamiltonian implementation of forced variational integrators. Thus, we focus on strong equivalence.

Strongly equivalent discrete triples $(L_d^1, f_d^{1\pm})$ and $(L_d^2, f_d^{2\pm})$ have equivalent forced discrete Legendre transforms, i.e., $\mathbb{F}^{f\pm}L_d^1 = \mathbb{F}^{f\pm}L_d^2$. By (2.2), this implies that

$$f_d^{1+}(q_k, q_{k+1}) - f_d^{2+}(q_k, q_{k+1}) = D_2L_d^2(q_k, q_{k+1}) - D_2L_d^1(q_k, q_{k+1})$$

and

$$f_d^{2-}(q_k, q_{k+1}) - f_d^{1-}(q_k, q_{k+1}) = D_1L_d^1(q_k, q_{k+1}) - D_1L_d^2(q_k, q_{k+1}),$$

so each difference in discrete forcing in two strongly equivalent triples must be integrable in at least one of the position variables. If we think of f_d^\pm as right and left discrete forces, then each discrete force difference must be integrable in its base-point variable, mirroring the continuous conclusion that $f_L^1(q, v) - f_L^2(q, v) = \nabla V^2(q) - \nabla V^1(q)$.

3.2.3 *Preserving continuous equivalence.* Theorems 3.1 and 3.2 provide a means of constructing forced variational integrators of known order by choosing quadrature rules and boundary-value solution methods. It is tempting to try to optimize the overall discretization by tailoring the discretization of L_d , f_d^+ and f_d^- , individually, but the resulting integrator is no longer well defined with respect to the equivalence relation defined above. The simplest way to generate an integrator that preserves continuous equivalence by our method is to choose the same quadrature rule and boundary-value solution method for all three discrete quantities.

In this case we have

$$L_d(q_0, q_1; h) = h \sum_{i=0}^n b_i L(q^i, v^i)$$

and

$$f_d^+(q_0, q_1; h) = h \sum_{i=0}^n b_i f_L(q^i, v^i) \frac{\partial q^i}{\partial q_1},$$

$$f_d^-(q_0, q_1; h) = h \sum_{i=0}^n b_i f_L(q^i, v^i) \frac{\partial q^i}{\partial q_0}.$$

Suppose (L^1, f_L^1) and (L^2, f_L^2) are equivalent, so

$$f_L^1(q, \dot{q}) - f_L^2(q, \dot{q}) = \nabla V^1(q) - \nabla V^2(q).$$

Constructing $(L_d^1, f_d^{1\pm})$ and $(L_d^2, f_d^{2\pm})$ from (L^1, f_L^1) and (L^2, f_L^2) we then have

$$\begin{aligned} f_d^{1+}(q_0, q_1; h) - f_d^{2+}(q_0, q_1; h) &= h \sum_{i=0}^n b_i (f_L^1(q^i, v^i) - f_L^2(q^i, v^i)) \frac{\partial q^i}{\partial q_1} \\ &= h \sum_{i=0}^n b_i (\nabla V^2(q^i) - \nabla V^1(q^i)) \frac{\partial q^i}{\partial q_1} \\ &= D_2 L_d^2(q_0, q_1; h) - D_2 L_d^1(q_0, q_1; h). \end{aligned}$$

Similarly, $f_d^{2-}(q_0, q_1; h) - f_d^{1-}(q_0, q_1; h) = D_1 L_d^1(q_0, q_1; h) - D_1 L_d^2(q_0, q_1; h)$. Thus, $(L_d^1, f_d^{1\pm})$ and $(L_d^2, f_d^{2\pm})$ are strongly equivalent.

3.3 Numerical examples

We construct our implementations by choosing a quadrature rule and a one-step map, using shooting to solve the boundary-value problem, which is analogous to the shooting-based variational integrator proposed in [Leok & Shingel \(2012\)](#). Quadrature of order q and a one-step map of order p produce an integrator of order $\min(p, q)$ since the shooting solution is order $p - 1$.

Table 1 shows that this construction converges as predicted. The second-order integrator uses trapezoidal quadrature and a second-order Taylor method shooting solution. The fourth-order integrator uses Simpson's rule quadrature and a fourth-order Taylor method shooting solution. Both were run on a damped harmonic oscillator with mass $m = 1$, spring constant $k = 1$ and damping coefficient $c = 0.01$. The table shows the effect of doubling the number of time steps per period T on the error after five periods. In both cases, the error decreases at the predicted rate.

TABLE 1 *Fourth- and second-order forced integrators show the expected convergence rates. The second-order integrator uses trapezoidal quadrature and second-order Taylor shooting. The fourth-order integrator uses Simpson's rule quadrature and fourth-order Taylor shooting. Both methods simulate a damped harmonic oscillator with $m = k = 1$ and damping coefficient $c = 0.01$*

Steps per period	20	40	80	160
Second-order error at $t = 5T$	0.2275	0.0557	0.0138	0.0035
Error ratio	–	4.0844	4.0362	3.9429
Fourth-order error at $t = 5T$	0.6551×10^{-4}	0.0516×10^{-4}	0.0034×10^{-4}	0.0002×10^{-4}
Error ratio	–	12.696	15.176	17

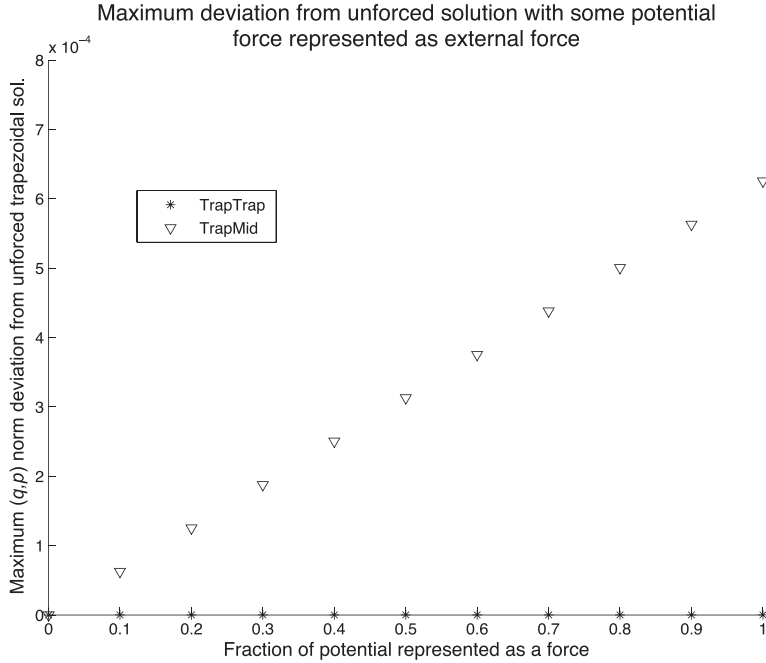


FIG. 1. As the system’s potential is increasingly represented as an external force, the non-equivalence-preserving construction produces solutions that veer further and further from the solution of an unforced representation. The equivalence-preserving construction produces the same solution whether the potential force is represented entirely through a potential within the Lagrangian or entirely as an external force.

Figure 1 illustrates the unpredictability of the results of non-equivalence-preserving discretizations. Two implementations use a second-order Taylor method shooting as the boundary-value solution method for both L_d and f_d^\pm with trapezoidal and/or midpoint quadrature. The equivalence-preserving discretization uses trapezoidal quadrature on both L_d and f_d^\pm . The non-equivalence-preserving implementation uses trapezoidal quadrature on L_d and midpoint quadrature on f_d^\pm . Both integrators were run on a conservative harmonic oscillator 11 times, with the potential force increasingly represented as an external force, i.e.,

$$L(q, v) = \frac{1}{2} v^T M v - (1 - \alpha) V(q) \quad f(q, v) = -\alpha \nabla V(q)$$

for $\alpha = 0, 0.1, 0.2, \dots, 1$. The step size was $h = 0.05$ for all runs. All values of α produce the same continuous equations of motion. The equivalence-preserving discretization produces the same solution regardless of α . The non-equivalence-preserving discretization produces α -dependent solutions that veer away from the unforced representation’s solution as α increases. We also compared a midpoint–midpoint construction to a midpoint–trapezoidal construction in the same way. The results lie directly atop the TrapTrap vs. TrapMid results shown. This unpredictability of results makes non-equivalence-preserving integrators unsuited for applications like interconnected systems in which a system’s forced representation will purposely be modified by modeling the component subsystems individually, with internal forces to account for the interaction with other subsystems.

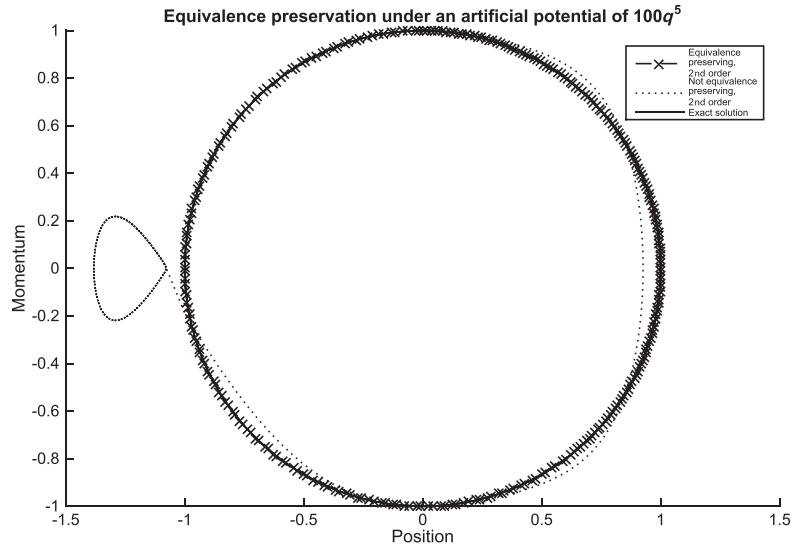


FIG. 2. Two integrators simulate a system whose equations of motion simplify to an undamped harmonic oscillator. An artificial potential of $100q^5$ was added to the Lagrangian, with the artificial potential force canceled by an external force. The second-order equivalence-preserving discretization is indistinguishable from the exact solution, while the second-order non-equivalence-preserving discretization veers wildly away.

Figure 2 shows a different example where failure to preserve equivalence produces wildly unpredictable results. We constructed a shooting-based variational integrator with fourth-order Runge–Kutta as the underlying one-step method and used the trapezoidal rule on L_d and f_d^\pm to construct a second-order equivalence-preserving integrator and Simpson’s rule on L_d and trapezoidal rule on f_d^\pm to construct a second-order non-equivalence-preserving integrator. Both of these methods were applied to simulate a system whose equations of motion simplify to an unforced harmonic oscillator. We inserted an artificial potential of $100q^5$, then canceled that potential force with a forcing function. As can be seen from the figure, the non-equivalence-preserving formulation fails to capture the motion and exhibits a qualitatively different behavior. The example is artificial but it highlights the need to preserve equivalence in developing a well-defined integrator.

4. Dirac variational integrators with forces

We now lay out the generalization of Dirac variational integrators to include forces. We begin from the variational perspective, where we can directly apply the ideas of forced variational integrators. We then formulate the resulting integrators from the perspective of discrete Dirac structures and finish with a basic numerical implementation.

4.1 Variational formulation

4.1.1 (+)-discrete Dirac mechanics. In Leok & Ohsawa (2011), the discrete variational principle for a Dirac system without external forces is the (+)-discrete Lagrange–d’Alembert–Pontryagin principle,

given by

$$\delta \sum_{k=0}^{N-1} [L_d(q_k, q_k^+) + p_{k+1}(q_{k+1} - q_k^+)] = 0,$$

where the variations vanish at the endpoints and are constrained to lie on a constraint distribution and the pairs of consecutive points in the discrete solution lie in a discrete constraint distribution. We define the forced principle in direct analogy to (2.1) as

$$\delta \sum_{k=0}^{N-1} [L_d(q_k, q_k^+) + p_{k+1}(q_{k+1} - q_k^+)] + \sum_{k=0}^{N-1} [f_d^-(q_k, q_k^+) \cdot \delta q_k + f_d^+(q_k, q_k^+) \cdot \delta q_k^+] = 0, \quad (4.1)$$

where the second sum represents the virtual work associated with the external forces. We have chosen the arguments of f_d^\pm to match those of L_d and we constrain the variations in the same way as Leok & Ohsawa (2011). Thus, δp_k and δq_k^+ are arbitrary and $\delta q_0 = \delta q_n = 0$. We impose $\delta q_k \in \Delta_Q(q_k)$ on the remaining δq_k after taking variations inside the sum and we insist that $(q_k, q_{k+1}) \in \Delta_Q^{d+}$. For the forced principle (4.1), these constraints yield

$$\begin{aligned} 0 &= \omega_{d+}^a(q_k, q_{k+1}), \\ q_{k+1} &= q_k^+, \\ p_{k+1} &= D_2 L_d(q_k, q_k^+) + f_d^+(q_k, q_k^+), \\ p_k &= -D_1 L_d(q_k, q_k^+) - f_d^-(q_k, q_k^+) + \mu_a \omega^a(q_k). \end{aligned}$$

These equations recover the (+)-discrete Lagrange–Dirac equations of Leok & Ohsawa (2011) when $f_d^\pm = 0$ and the forced DEL equations (Marsden & West, 2001) in the absence of constraints. When both the forces and constraints are zero, we recover the classic DEL equations.

4.1.2 (–)-discrete Dirac mechanics. In Leok & Ohsawa (2011), (–)-discrete Dirac mechanics derives from the (–)-discrete Lagrange–d’Alembert principle,

$$\delta \sum_{k=0}^{N-1} [L_d(q_{k+1}^-, q_{k+1}) - p_k(q_k - q_{k+1}^-)] = 0,$$

with constraints on the variations, and the pairs of sequential points as before. We add forces to this principle as

$$\delta \sum_{k=0}^{N-1} [L_d(q_{k+1}^-, q_{k+1}) - p_k(q_k - q_{k+1}^-)] + \sum_{k=0}^{N-1} [f_d^-(q_{k+1}^-, q_{k+1}) \cdot \delta q_{k+1}^- + f_d^+(q_{k+1}^-, q_{k+1}) \cdot \delta q_{k+1}] = 0, \quad (4.2)$$

again choosing the arguments of f_d^\pm to match those of L_d and constraining the variations as in [Leok & Ohsawa \(2011\)](#). In this case then, δp_k and δq_k^- are arbitrary and $\delta q_0 = \delta q_N = 0$. We impose $\delta q_k \in \Delta_Q(q_k)$ on the remaining δq_k after taking variations inside the sums and we insist that $(q_k, q_{k+1}) \in \Delta_Q^{d-}$. With these conditions, computing variations of the principle (4.2) yields

$$\begin{aligned} 0 &= \omega_{d-}^a(q_k, q_{k+1}), \\ q_k &= q_{k+1}^-, \\ p_k &= -D_1 L_d(q_{k+1}^-, q_{k+1}) - f_d^-(q_{k+1}^-, q_{k+1}), \\ p_{k+1} &= D_2 L_d(q_{k+1}^-, q_{k+1}) + f_d^+(q_{k+1}^-, q_{k+1}) + \mu_a \omega^a(q_{k+1}). \end{aligned}$$

In the unforced case, these equations recover the $(-)$ -discrete Dirac equations of [Leok & Ohsawa \(2011\)](#). In the unconstrained case, we recover the forced DEL equations of [Marsden & West \(2001\)](#). In the absence of both, we again recover the classic DEL equations.

4.2 Discrete Dirac structure formulation

The dynamics of an unforced, continuous Lagrange–Dirac dynamical system can be expressed in terms of Dirac structures as $(X, \mathfrak{D}L) \in D$, where X is the partial vector field of the motion, $\mathfrak{D}L$ the Dirac differential of the Lagrangian L and D a Dirac structure on T^*Q . To accommodate forces in the continuous case the force $F : TQ \rightarrow T^*Q$ is lifted to a map $\tilde{F} : TQ \rightarrow T^*T^*Q$ such that

$$\langle \tilde{F}(q, v), w \rangle = \langle F(q, v), T\pi_Q(w) \rangle.$$

Locally, \tilde{F} is given by $\tilde{F}(q, v) = (q, p, F(q, v), 0)$. The forced Lagrange–Dirac dynamical system is then given by

$$(X, \mathfrak{D}L(q, v) - \tilde{F}(q, v)) \in D_{\Delta_Q}(q, p). \quad (4.3)$$

Without forces, we can express $(+)$ - and $(-)$ -discrete Dirac mechanics in terms of discrete Dirac structures as

$$(X_d^k, \mathfrak{D}^+ L_d(q_k, q_k^+)) \in D_{\Delta_Q}^{d+}$$

and

$$(X_d^k, \mathfrak{D}^- L_d(q_{k+1}^-, q_{k+1})) \in D_{\Delta_Q}^{d-},$$

where

$$D_{\Delta_Q}^{d+} \equiv \left\{ ((z, z^+), \alpha_z) \in (T^*Q \times T^*Q) \times T^*(Q \times Q^*) \mid (z, z^+) \in \Delta_{T^*Q}^{d+}, \alpha_z - \Omega_{d+}^b(z, z^+) \in \Delta_{Q^* \times Q^*}^{\circ} \right\}$$

and

$$D_{\Delta_Q}^{d-} \equiv \left\{ ((z^-, z), \alpha_z) \in (T^*Q \times T^*Q) \times T^*(Q^* \times Q) \mid (z^-, z) \in \Delta_{T^*Q}^{d-}, \alpha_z - \Omega_{d-}^b(z^-, z) \in \Delta_{Q^* \times Q}^{\circ} \right\},$$

which amount to the conditions

$$\begin{aligned} (q_k, D_2 L_d, -D_1 L_d, q_k^+) - (q_k, p_{k+1}, p_k, q_{k+1}) &\in \Delta_{Q \times Q^*}^\circ, \\ (-D_1 L_d, q_{k+1}, -q_{k+1}^-, -D_2 L_d) - (p_k, q_{k+1}, -q_k, -p_{k+1}) &\in \Delta_{Q^* \times Q}^\circ. \end{aligned}$$

To introduce forces into these expressions in a manner that is consistent with the discrete equations of motion that are derived variationally, we require

$$(q_k, D_2 L_d + f_d^+, -D_1 L_d - f_d^-, q_k^+) - (q_k, p_{k+1}, p_k, q_{k+1}) \in \Delta_{Q \times Q^*}^\circ, \quad (4.4a)$$

$$(-D_1 L_d - f_d^-, q_{k+1}, -q_{k+1}^-, -D_2 L_d - f_d^+) - (p_k, q_{k+1}, -q_k, -p_{k+1}) \in \Delta_{Q^* \times Q}^\circ. \quad (4.4b)$$

To this end, define \tilde{F}_{d+} and \tilde{F}_{d-} as

$$\begin{aligned} \tilde{F}_{d+}(q_k, q_k^+) &= (0, -f_d^+(q_k, q_k^+), f_d^-(q_k, q_k^+), 0), \\ \tilde{F}_{d-}(q_{k+1}^-, q_{k+1}) &= (f_d^-(q_{k+1}^-, q_{k+1}), 0, 0, f_d^+(q_{k+1}^-, q_{k+1})). \end{aligned}$$

Then we can write condition (4.4) as

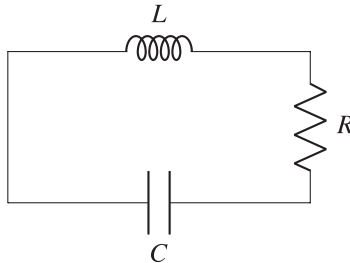
$$(X_d^k, \mathfrak{D}^+ L_d(q_k, q_k^+) - \tilde{F}_{d+}(q_k, q_k^+)) \in D_{\Delta Q}^{d+}, \quad (4.5a)$$

$$(X_d^k, \mathfrak{D}^- L_d(q_{k+1}^-, q_{k+1}) - \tilde{F}_{d-}(q_{k+1}^-, q_{k+1})) \in D_{\Delta Q}^{d-}, \quad (4.5b)$$

mimicking the continuous expression (4.3). In (4.5) we regard $\mathfrak{D}^\pm L_d - \tilde{F}_{d\pm}$ as an abstract expression and impose base-point matching after the subtraction.

4.3 Numerical example

We close with a simple implementation of the (+)-forced discrete Dirac equations. We simulate a basic resistor-inductor-capacitor (RLC) resonator with a single resistor, inductor and capacitor. We assume some existing voltage in the system but no replenishing voltage source.



The system is described as a Lagrange–Dirac system in terms of the charges and currents in each component, $(q^C, q^L, q^R, i^C, i^L, i^R)$. Let L, R and C denote the inductance, resistance and capacitance of the components. Then the system has Lagrangian function

$$L(q, i) = \frac{L}{2}(i^L)^2 - \frac{(q^C)^2}{2C}$$

and force

$$f(q, i) = -i^R R.$$

Kirchhoff's current law imposes the following constraints:

$$\begin{aligned} i^L - i^R &= 0, \\ i^R - i^C &= 0. \end{aligned}$$

Thus, our constraints can be described in terms of the annihilator distribution $\Delta_Q^\circ = \text{span}\{\omega^1, \omega^2\}$ for $\omega^1 = dq^L - dq^R$ and $\omega^2 = dq^R - dq^C$. We follow the discretization in [Leok & Ohsawa \(2011\)](#) to construct $\omega_{d+}^{1,2}$ as $\omega_{d+}^a(q_k, q_{k+1}) = \omega^a(q_k, \mathcal{R}_{q_k}^{-1}(q_{k+1}))$ for $\mathcal{R}_q(v) = q + vh$.

Existence of exact discrete quantities in the Dirac setting has not yet been established so we cannot turn to them to guide the discretization for L_d and f_d^\pm . Establishing such quantities is obviously desirable and a topic for future work. For now we choose a simple L_d and f_d^\pm shown to work well in both the forced, non-Dirac setting ([Marsden & West, 2001](#)) as well as the nonholonomic integrator setting ([Cortés & Martinez, 2001](#)). Namely, we choose L_d and f_d^\pm to be

$$L_d^{1/2}(q_k, q_k^+) = hL \left(\frac{q_k^+ + q_k}{2}, \frac{q_k^+ - q_k}{h} \right)$$

and

$$f_d^{1/2\pm} = \frac{h}{2} f \left(\frac{q_k^+ + q_k}{2}, \frac{q_k^+ - q_k}{h} \right).$$

[Marsden & West \(2001\)](#) introduces these forces as the natural complement to $L_d^{1/2}$. The discretizations $L_d^{1/2}$ and $f_d^{1/2\pm}$ both correspond to choosing midpoint quadrature and a linear boundary-value solution to approximate L_d^E and $f_d^{E\pm}$ in the forced, non-Dirac setting. Thus, the set $(L_d^{1/2}, f_d^{1/2\pm})$ is natural and equivalence preserving by our earlier analysis as well.

Figure 3 shows that the simulated charge in the capacitor over time (dotted line) is almost identical to the exact solution (solid line) using a time step of $h = 0.05$ for 1000 time steps.

5. Conclusions and future work

In this work we have successfully integrated the presence of external forcing into the discrete Dirac framework originally presented in [Leok & Ohsawa \(2011\)](#) so that our forced discrete Dirac integrators recover the forced DEL integrators of [Marsden & West \(2001\)](#). We have also studied the effects of

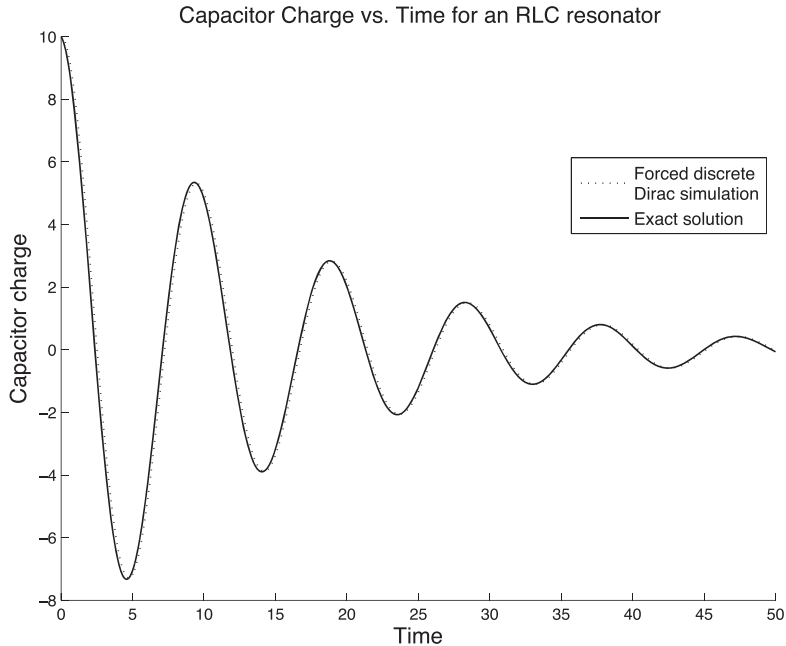


FIG. 3. A forced discrete Dirac integrator accurately simulates the decaying charge oscillations of a capacitor in an RLC resonator. The resonator consists of a single loop with one inductor ($L = 0.75$), one capacitor ($C = 3$) and one resistor ($R = 0.1$). The step size is $h = 0.05$.

shifting forced representations on the accuracy of numerical simulations and presented a straightforward method for constructing integrators that avoid these spurious effects.

The work is motivated by the applications to the interconnection of discrete Dirac mechanics. Continuous interconnections can be equivalently represented as either constraints or interaction forces acting between subsystems. One would hope to obtain a similar equivalence at the discrete level. In recent work (Parks & Leok, 2017) we have derived discrete interconnections from the point of view of constraints. However, we do not yet have an equivalent representation in terms of discrete interaction forces. There is a lack of equivalence between forced and constrained representations throughout the discrete literature, though recent work has rectified this issue on vectors spaces through Hamel's formalism (Ball & Zenkov, 2015).

The development of a discrete theory of Dirac interconnected systems based on interaction forces is desirable, as it is better adapted than the constraint-based approach to the construction of higher-order discretizations. In essence, the limitation of the constraint-based approach is that the high-order approximation of a continuous nonholonomic constraint depends on the choice of the discrete Lagrangian and its associated discrete solution trajectory, which complicates its construction and analysis.

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