THE FUNDAMENTAL THEOREM OF ALGEBRA

The fundamental theorem of algebra states the following:

Theorem 0.1.
Let $c_0, c_1, \dots, c_k \in \mathbb{C}$ with $k \geq 1$ be a finite sequence of complex numbers. Then the polynomial

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k$$

can be written as

$$p(x) = a_0(x - a_1)(x - a_2) \cdot \cdots \cdot (x - a_k),$$

where $a_0, a_1, a_2, \dots, a_k \in \mathbb{C}$ are a finite sequence of complex numbers that depend on $c_0, c_1, \dots, c_k$ and that are unique up to reordering.

These complex numbers $a_1, a_2, \dots, a_k$ are precisely the roots of p. In other words, if $x \in \mathbb{C}$ with $p(x) = 0$, then x is one of the $a_1, \dots, a_k$.

Remark 0.2.
Polynomials of the form $x - a$ for some $a \in \mathbb{C}$ are linear functions. In the context of polynomial factorization, they are called *linear factors*.

Remark 0.3.
It is important that $k \geq 1$, which means that we exclude that case $k = 0$. Indeed, if $k = 0$, then the polynomial $p(x) = c_0$ is just a constant function. If $c_0 \neq 0$, then this constant function does not have a root, and if $c_0 = 0$, then every $x \in \mathbb{C}$ is a root of this polynomial.

Example 0.4.
The following polynomials illustrate the theorem:

$$\begin{aligned}
p_1(x) &= 2x^2 - 18 = 2(x-3)(x+3), \\
p_2(x) &= x^3 - x^2 - 4x + 4 = (x-2)(x+2)(x-1), \\
p_3(x) &= \mathbf{i}x^2 + \mathbf{i} = \mathbf{i}(x - \mathbf{i})(x + \mathbf{i}), \\
p_4(x) &= x^3 + (-2 + \mathbf{i})x^2 + (-1 - 3\mathbf{i})x + 2(1 + \mathbf{i}) = (x - 1 - \mathbf{i})(x-2)(x-1).
\end{aligned}$$

Problem 1.
Can you tell the constant factor a_0 from the coefficients $c_0, c_1, \dots, c_k$ immediately?

Problem 2.
Let p and q be polynomials of degree $k \in \mathbb{N}$. Explain the following: if p has the same roots as q, then there exists $a \in \mathbb{C}$ with $p = aq$.