MATH 102 – HOMEWORK ASSIGNMENT 2

Due Friday, October 13th, 2017 before the beginning of the lecture.
Handwritten submissions only.

Exercise 1 (4 points)**.**
Let $\alpha \in \mathbb{R}$. Consider the matrix $A \in \mathbb{R}^{3\times 3}$ and the vector $b \in \mathbb{R}^3$ given by

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 3 \\ 2 & -2 & \alpha \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

For which values of α does the linear system of equations

$$Ax = b$$

have a unique solution $x \in \mathbb{R}^3$? Assuming that such a unique solution exists, write down the solution $x \in \mathbb{R}^3$ in terms of α, the entries of A, and the entries of b.

Exercise 2 (4 points)**.**
Give an example for the following statements.

(a) There exist non-zero matrices $A, B \in \mathbb{R}^{2\times 2}$ such that $AB \in \mathbb{R}^{2\times 2}$ is zero.
(b) There exist non-zero matrices $A, B, C \in \mathbb{R}^{2\times 2}$ such that $AC = BC$ and $A \neq B$.
(c) There exist nonsingular matrices $A, B \in \mathbb{R}^{2\times 2}$ such that $A + B$ is singular.
(d) There exist $A \in \mathbb{R}^{2\times 3}$ and $B \in \mathbb{R}^{3\times 2}$ such that $AB = \mathrm{Id}_2$ and $BA \neq \mathrm{Id}_3$.

Exercise 3 (4 points)**.**
Let $A \in \mathbb{R}^{n\times n}$ and $k \in \mathbb{N}$ such that

$$A^{k+1} = 0.$$

Compute the matrix products

$$C_1 = (\mathrm{Id}_n - A)\left(\mathrm{Id}_n + A + \cdots + A^k\right), \quad C_2 = \left(\mathrm{Id}_n + A + \cdots + A^k\right)(\mathrm{Id}_n - A).$$

Prove that $\mathrm{Id}_n - A$ is nonsingular.

Exercise 4 (4 points)**.**
There are many different methods of solving linear systems of equations on a computer; Gaussian elimination is only the most prominent of them.

A huge mathematical literature exists on so-called *iterative methods*, which have a completely different philosophy than Gaussian elimination: the method starts with an (arbitrary) guess for the solution of the linear system and then iteratively improves this first initial guess by repeating a computational scheme over and over again.

The actual solution of the linear system of equations typically never appears in that process of computation, but after repeating the computional scheme often enough, we usually have a good guess of the solution. In many applications (but not all) this is already enough. The advantage of iterative methods is that they require much less memory and computing time when the system of equations is very large. The disadvantage is that they only work for matrices with special properties.

Consider the matrix $A \in \mathbb{R}^{2\times 2}$ and the vector $b \in \mathbb{R}^2$ with

$$A = \begin{pmatrix} 0.5 & -0.1 \\ 0.2 & 0.6 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Compute the exact solution $x \in \mathbb{R}^2$ of the linear system of equations

$$Ax = b.$$

For such a system we can easily derive the exact solution by hand, but we try out an iterative scheme instead. Let $x_0 = 0 \in \mathbb{R}^2$ be the zero vector. Compute the following vectors:

$$\begin{aligned} x_1 &:= x_0 + (b - Ax_0)\,, \\ x_2 &:= x_1 + (b - Ax_1)\,, \\ x_3 &:= x_2 + (b - Ax_2)\,, \\ x_4 &:= x_3 + (b - Ax_3)\,. \end{aligned}$$

Write down what you observe.