**MATH 102 – HOMEWORK ASSIGNMENT 3**

*Due Friday, October 20th, 2017 before the first mideterm.*

*Handwritten submissions only.*

**Exercise 1** (2+2 points)**.**
Let $A, B, C \in \mathbb{R}^{m\times n}$ be matrices. Show the following statements:

(a) If $A$ is row equivalent to $B$, then $B$ is row equivalent to $A$.
(b) If $A$ is row equivalent to $B$ and $B$ is row equivalent to $C$, then $A$ is row equivalent to $C$.

**Exercise 2** (1+1+1+1 points)**.**
Bring the following complex numbers into the form $a + b\mathbf{i}$:

(a)
$$(5+3\mathbf{i})(6+3\mathbf{i})$$
(b)
$$(1+\mathbf{i})(1-\mathbf{i})$$
(c)
$$\mathbf{i}\cdot(2+\mathbf{i})^{-1}$$
(d)
$$1\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\mathbf{i}\right)^2$$

**Exercise 3** (1+1+1+1 points)**.**
Bring the following complex numbers into the form $a + b\mathbf{i}$ and compute their absolute value:

(a)
$$\frac{1}{\mathbf{i}}+\frac{1}{1+\mathbf{i}}+\frac{1}{1-\mathbf{i}}$$
(b)
$$\frac{5-3\mathbf{i}}{\mathbf{i}}+(1+2\mathbf{i})(1-2\mathbf{i})$$
(c)
$$(\mathbf{i})^8$$
(d)
$$\left(1+\sqrt{2}\mathbf{i}\right)(2-3\mathbf{i})(3+\mathbf{i})$$

**Exercise 4** (4 points)**.**
Consider the complex number
$$z=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathbf{i}.$$

(a) Compute the powers $z^2$, $z^3$, $z^4$, $z^5$. Reminder: $z^2 = z\cdot z$, and $z^3 = z\cdot z\cdot z$, and so on.
(b) Draw (schematically) the complex numbers $z, z^2, z^3, z^4, z^5$ as vectors in $\mathbb{R}^2$. Can you guess the power $z^6$ by looking at that drawing?

*Completing the following bonus exercise is completely voluntary. The points of the bonus exercise can contribute to your score, but they are not part of the maximal point number.*

**Exercise 5** (Bonus Exercise: Complex Numbers in Quantum Computing, 4 points)**.**
In the classical theory of computation, the units of information are bits. Mathematically, a bit may be represented by a member in the set $\{0, 1\}$, and physically, a bit may be realized any device which distinguishes the states *on* or *off*.

In the theory of quantum computation, however, the units of information are so-called *qubits*, which is short for *quantum bit*. Mathematically, a qubit is a pair of complex numbers $(\alpha, \beta) \in \mathbb{C}^2$ with the property

$$|\alpha|^2 + |\beta|^2 = 1.$$

Physically, a qubit may be represented by any quantum mechanical system that is in superposition between exactly two states. For example, a qubit can be physically realized by a rotating atom that is in quantum superposition of *rotating left* and *rotating right*. Another example could be Schrödinger's cat, being in quantum superposition between *dead* and *alive*.

The basic computing operations that a quantum computer performs on single qubit are mathematically formalized by a class of matrices with complex entries, called *unitary matrices*. Accordingly, applying such a basic computational operations on a single qubit is mathematically represented by multiplying the qubit, seen as a 2-vector with complex entries, by such $2 \times 2$ matrix with complex entries.

(i) Verify that the following pair $v \in \mathbb{C}^2$ of complex numbers represents a qubit:

$$v = \begin{pmatrix} \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\mathbf{i} \\ \\ \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}\mathbf{i} \end{pmatrix}.$$

(ii) Compute the following matrix vector product over the field of complex numbers:

$$\begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{i}}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-\mathbf{i}}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

(iii) Verify that the result $(\gamma, \delta) \in \mathbb{C}^2$ represents a qubit.