MATH 102 – HOMEWORK ASSIGNMENT 7

Due Friday, November 17th, 2017 before the lecture.

Handwritten submissions only.

Exercise 1 (4 points)**.**
We conduct Gaussian elimination and the LU decomposition over the field of complex numbers.

(1) Bring the following 2×2 system $A \cdot z = b$ of linear equations over the field $\mathbb{C}$ into triangular form:

$$\begin{pmatrix} 2+\mathbf{i} & 3\mathbf{i} \\ -1-\mathbf{i} & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1+\mathbf{i} \\ 1-\mathbf{i} \end{pmatrix}.$$

(2) Compute the solution $z = (z_1, z_2) \in \mathbb{C}^2$ of the system above.
(3) Determine a lower triangular matrix $L' \in \mathbb{C}^{2\times 2}$ such that $L'A$ is upper triangular.
(4) Determine a lower triangular matrix $L \in \mathbb{C}^{2\times 2}$ and an upper triangular matrix $U \in \mathbb{C}^{2\times 2}$ such that $A = LU$.

Exercise 2 (4 points)**.**
Let $A \in \mathbb{R}^{n\times n}$ be an upper triangular matrix. Show that the determinant of A is the product of the diagonal entries, i.e.,

$$\det(A) = \prod_{i=1}^{n} a_{ii}.$$

Hint: Use induction.

Exercise 3 (4 points)**.**
Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and let $\pi \in \Pi(1, n)$ be a permutation. We let $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ be the $n \times n$ matrix whose i-th row is the $\pi(i)$-th row of A. This means that

$$b_{ij} = a_{\pi(i),j}$$

for $1 \leq i, j \leq n$. Show that

$$\det(B) = \operatorname{sgn}(\pi) \cdot \det(A).$$

Hint: use the Leibniz formula.

Exercise 4 (1+1+2 points)**.**
For each $\pi \in \Pi(1, n)$ we define the permutation matrix $P_\pi = (p_{ij}) \in \mathbb{R}^{n\times n}$ by the property

$$p_{ij} = \begin{cases} 1 & \text{if } \pi(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

(1) Show that for every n-dimensional vector with entries $v_1, \ldots, v_n \in \mathbb{R}$ we have

$$P_\pi \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_{\pi(1)} \\ v_{\pi(2)} \\ \vdots \\ v_{\pi(n)} \end{pmatrix}$$

(2) Show that $\det(P_\pi) = \operatorname{sgn}(\pi)$.
(3) Show that for $\pi, \rho \in \Pi(1, n)$ we have

$$P_\rho \circ P_\pi = P_{\pi\circ\rho}$$

Bonus Exercise 1 (1+1+4 points)**.**
Apart from abstract linear algebra and the solution theory of systems of linear equations, determinants have a geometric interpretation: they signify the *signed volume* of a parallelepepid. Let us take a look at different dimensions:

For $n = 1$, the parallelepepids are just line segments. The determinant $\det(A)$ of a 1×1 matrix $A = (a_{11})$ is just the (signed) length of a one-dimensional line segment.

For $n = 2$, the parallelepepid of two vectors $a, b \in \mathbb{R}^2$ is the set

$$P(a, b) = \{ \alpha a + \beta b \mid \alpha, \beta \in [0, 1] \} .$$

The signed area of this set is the determinant

$$\mathrm{vol}(a, b) = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} .$$

(1) Draw the parallelepepid for $a_1 = 3, a_2 = 5$ and $b_1 = 4, b_2 = 7$.
(2) Show that $\mathrm{vol}(a, b) = 0$ if and only if there exists $\gamma \in \mathbb{R}$ such that $a = \gamma b$ or $b = \gamma a$. *Hint: make a case distinction whether some of the columns are zero.*

This definition of the signed area is motivated as follows: the area should not change if we rotate the geometric body around the origin, so after a rotation of the parallelepepid we may assume that $a_2 = 0$. Then $\mathrm{vol}(a, b) = a_1 b_2$. But this is just the length of the base of the parallepepid times its (signed) height. This is the area of the parallepepid according to Cavalieri's principle [i]: if we think of $P(a, b)$ of a stack of very thin layers of length b, then the volume of this stack remains the same when the shift the layers such that their ends match; but that's just a stack of base length b_2 and height a_1.

When $n \geq 3$, let $a_1, a_2, \ldots, a_n \in \mathbb{R}^n$. The n-dimensional parallelepepid spanned by these vectors is

$$P(a_1, \ldots, a_n) := \{ \alpha_1 a_1 + \cdots + \alpha_n a_n \mid \alpha_1, \ldots, \alpha_n \in [0, 1] \} .$$

Let $A \in \mathbb{R}^n$ be the matrix with columns $a_1, \ldots, a_n$. That $\det(A)$ should be the signed n-dimensional volume of $P(a_1, \ldots, a_n)$ is motivated as follows:

If $\det(A) = 0$, then the parallepepid collapses into a lower dimensional geometric object. It's n-dimensional volume should clearly be zero.

If $\det(A) \neq 0$, then A is invertible. Consequently, there exists a sequence of elementary row operations $E_1, \ldots, E_N \in \mathbb{R}^{n \times n}$ such that $A = E_N \cdot \cdots \cdot E_1 \cdot \mathrm{Id}_n$. Generally, we have

$$E_{i,j,\alpha} = E_{i,\alpha^{-1}} E_{i,j,1} E_{i,\alpha}$$

so we may assume that all Type III row operations have merely a scaling by 1. The effect of the row operations on the signed volume should be follows:

- If two base vectors of the parallepepid are switched, then the signed volume should switch signs.
- If the parallepepid is scaled in one coordinate direction by a factor $\alpha \neq 0$, then the its signed volume should scale by the same factor.
- Adding one row of A to another row corresponds to a shear mapping. [ii] Again argueing by Cavalieri's principle, the signed volume of the geometric body $P(a_1, \ldots, a_n)$ should not change under such a geometric transformation.

These thoughs suggest that $\det(A) = \det(E_N) \cdot \cdots \cdot \det(E_1) \det(\mathrm{Id}_n)$ should be the signed volume of the parallelepepid $P(a_1, \ldots, a_n)$.

(3) Decompose the following matrix A into the product of elementary matrices and find the signed volume of the corresponding parallelepepid:

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 2 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

[i] https://en.wikipedia.org/wiki/Cavalieri's_principle
[ii] https://en.wikipedia.org/wiki/Shear_mapping