MATH 102 – HOMEWORK ASSIGNMENT 7

Due December 1st, 2017 before the lecture.
Handwritten submissions only.

Exercise 1 (4 points)**.**
Let $A, B \in \mathbb{R}^{n\times n}$. Prove the following:
(1) $\det(A^k) = \det(A)^k$ for $k \in \mathbb{N}$.
(2) $\det(\alpha A) = \alpha^n \det(A)$
(3) $\det(-A) = (-1)^n \det(A)$.
(4) $\det(AB) = \det(BA)$

Exercise 2 (4 points)**.**
Let $A \in \mathbb{R}^n$ be invertible with $n \geq 2$. Prove that

$$\det(\operatorname{adj} A) = (\det(A))^{n-1}$$

Exercise 3 (4 points)**.**
Compute the following determinants:

$$A = \begin{pmatrix} 3 & -1 & 5 & 1 & 4 & 6 \\ 4 & -2 & 7 & -1 & 0 & 1 \\ 5 & 0 & 5 & 2 & 0 & 2 \\ 1 & 0 & 8 & 0 & 0 & -3 \\ -2 & 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ e & 0 & e^\pi & 4 & 5 & 1 & \sqrt{\pi} \\ e^2 & 1 & \frac{17}{31} & \sqrt{6} & \sqrt{7} & \sqrt{8} & \sqrt{10} \\ e^3 & 0 & -e & \pi & e & 0 & \pi^e \\ e^4 & 0 & 10001 & 0 & \pi^{-1} & 0 & e^2\pi \\ e^6 & 0 & \sqrt{2} & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 4 (4 points)**.**
Let W, X, Y, Z be vector spaces. Let $A \in L(W, X)$ and $B \in L(Y, Z)$. Consider the mapping

$$T : L(X, Y) \to L(W, Z), \quad C \mapsto B \circ C \circ A$$

Explain the mapping T in your own words (where does it map from, where does it map to, what does it do?) and show that

$$T \in L\Big(L(X, Y), L(W, Z)\Big).$$

Exercise 5 (4 points)**.**
For $r \geq 0$ we let $\mathcal{P}_r$ denote the vector space of polynomials of degree at most r over the interval $[0,1]$, so

$$\mathcal{P}_r := \left\{ p : [0,1] \to \mathbb{R} \;\middle|\; p(x) = \sum_{k=0}^{r} a_k x^k \text{ with } a_0, \ldots, a_r \in \mathbb{R} \text{ for all } x \in [0,1] \right\}.$$

(1) For $r \geq 1$ consider the derivative

$$\partial : \mathcal{P}_r \to \mathcal{P}_{r-1}, \quad \partial \left(\sum_{k=0}^{r} a_k x^k \right) \mapsto \sum_{k=0}^{r} k \cdot a_k x^{k-1}.$$

Show that this mapping is linear.

(2) What is the kernel of the derivative mapping?

(3) Show that integral mapping

$$\int : \mathcal{P}_r \to \mathbb{R}, \quad p \mapsto \int_0^1 p \, \mathrm{dx},$$

is linear.

(4) Explain why

$$\mathcal{P}_{r,0} := \left\{ p \in \mathcal{P}_r \;\middle|\; \int_0^1 p \, \mathrm{dx} = 0 \right\}.$$

is a subspace of $\mathcal{P}_r$.

Exercise 6 (4 points)**.**
Explain what the following linear mappings do geometrically and draw an illustrative picture:

$$T_1 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$T_2 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$T_3 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$T_4 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

Exercise 7 (4 points)**.**
Consider the following matrix of size $n \times n$:

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

Describe the effect of the matrix A when multiplied to a vector $v \in \mathbb{R}^n$ in terms of the entries of the resulting vector. Can you describe the actions of the matrices $A^2, A^3, \ldots$? Write down the matrix A^3 for the case $n = 5$.