MATH 102 – HOMEWORK ASSIGNMENT 9

Due December 8th, 2017 before the lecture.
Handwritten submissions only.

Exercise 1 (6 points)**.**
Let $v_1, \ldots, v_m \in \mathbb{R}^n$ be vectors and let $w_1, \ldots, w_m$ be obtained by one of the following three procedures:

- Switching vectors:

$$w_1 = v_1,\ \ldots, w_{i-1} = v_{i-1}, w_i = v_j, w_{i+1} = v_{i+1}, \ldots, w_{j-1} = v_{j-1}, w_j = v_i, w_{j+1} = v_{j+1}, \ldots, w_n = v_n$$

- Scaling the i-th vector by a nonzero factor $\alpha \in \mathbb{R}$:

$$w_1 = v_1, \ldots, w_{i-1} = v_{i-1}, w_i = \alpha v_i, w_{i+1} = v_{i+1}, \ldots, w_n = v_n.$$

- Adding the α-th multiple of v_j to the i-th vector:

$$w_1 = v_1, \ldots, w_{j-1} = v_{j-1}, w_j = v_j + \alpha v_j, w_{j+1} = v_{j+1}, \ldots w_n = v_n.$$

For each of these three constructions, show that $w_1, \ldots, w_m$ has the same span as $v_1, \ldots, v_m$ and show that $w_1, \ldots, w_m$ is linearly independent if and only if $v_1, \ldots, v_m$ is.

Exercise 2 (2 points)**.**
Let V be a vector space and let $v_1, \ldots, v_m \in V$ be vectors. Show that if they are linearly dependent if and only if there exists $1 \leq i \leq m$ such that v_i is a linear combination of $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m$.

Exercise 3 (6 points)**.**
Find bases for the kernels and the ranges of the following matrices.

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 2 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -4 & 2 & 0 & 0 & 0 \\ 5 & -1 & 1 & 3 & 7 \\ -4 & 2 & 0 & 8 & 0 \\ 13 & 0 & 0 & 0 & 0 \\ 1 & -1 & 4 & 0 & 0 \end{pmatrix}.$$

Exercise 4 (2 points)**.**
Assume that $v_1, \ldots, v_n \in \mathbb{R}^n$ and $w_1, \ldots, w_n \in \mathbb{R}^n$ are two different bases of $\mathbb{R}^n$, and that they are also the columns of the matrices $A, B \in \mathbb{R}^{n \times n}$:

$$A = \left(\begin{array}{c|c|c|c} | & | & & | \\ v_1 & v_2 & \ldots & v_n \\ | & | & & | \end{array} \right), \quad B = \left(\begin{array}{c|c|c|c} | & | & & | \\ w_1 & w_2 & \ldots & w_n \\ | & | & & | \end{array} \right).$$

Prove that the basis transformation from the basis $v_1, \ldots, v_n$ to the basis $w_1, \ldots, w_n$ is given by the matrix $B^{-1}A$. In other words, whenever $x \in \mathbb{R}^n$ is a vector with the basis representations

$$x = \alpha_1 v_1 + \cdots + \alpha_n v_n = \beta_1 w_1 + \cdots + \beta_n w_n,$$

then

$$B^{-1} A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}.$$