Math 102 – Fall Quarter 2017 – Midterm I

Full name: ______________________________

Student ID: ______________________________

Instructions:

(1) Please print your full name and your student ID.

(2) Using cheatsheets, calculators, books, or phones is **not** allowed.

(3) You have 50 minutes to complete the test.

Problem	Points
1	
2	
3	
4	
5	
6	
Σ	

Problem 1 (10 points)**.**
To each statement, write TRUE or FALSE, depending on whether the statement is true or false.

(1) The $n \times n$ identity matrix Id_n has all entries 1.

(2) The following matrix–matrix product is computed correctly:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(3) If $A \in \mathbb{R}^{2\times 2}$ with $A^2 = 0$, then $A = 0$.

(4) The following matrix $A \in \mathbb{R}^{2\times 2}$ is invertible for all values $\alpha \in \mathbb{R}$:

$$A = \begin{pmatrix} 5 & 0 \\ 0 & \alpha \end{pmatrix}.$$

(5) The following matrix $B \in \mathbb{R}^{2\times 2}$ is invertible for all values $\beta \in \mathbb{R}$:

$$B = \begin{pmatrix} 5 & \beta \\ 0 & 5 \end{pmatrix}.$$

(6) For all matrices $A, B \in \mathbb{R}^{n\times n}$ we have $AB = BA$.

(7) For invertible matrices $A, B \in \mathbb{R}^{n\times n}$ we have $AB = BA$.

(8) For all invertible matrices $A, B \in \mathbb{R}^{n\times n}$ the product AB is invertible with $(AB)^{-1} = A^{-1}B^{-1}$.

(9) The following matrix–matrix product is computed correctly:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}.$$

(10) The following matrix–matrix product is computed correctly:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -5 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & 5 \end{pmatrix}.$$

Problem 2 (8 points).
For each statement, write TRUE or FALSE, depending on whether the statement is true or false.
If it is false, then write down a counterexample. No proof is necessary.

(1) For every field $\mathbb{F}$ and every $y \in \mathbb{F}$ there exists $x \in \mathbb{F}$ with $x^2 = y$.

(2) For every field $\mathbb{F}$ and all $a, b, c \in \mathbb{F}$ we have $a(b + c) = ab + ac$.

(3) For every field $\mathbb{F}$ and all $a, b, c \in \mathbb{F}$ we have $(a + b)c = ac + bc$.

(4) For all $A, B \in \mathbb{R}^{3\times 3}$ we have $A + B = B + A$.

(5) If $A, B \in \mathbb{R}^{2\times 2}$ are upper triangular, then their product AB is upper triangular.

(6) All diagonal matrices are invertible.

(7) Every real number $a \in \mathbb{R}$ can be represented as $a = p/q$ with $p, q \in \mathbb{Z}$ and $q \neq 0$.

(8) If $A \in \mathbb{R}^{2\times 2}$ is invertible, then $AA^{-1}A = A$.

Problem 3 (10 points)**.**
For each the following pairs of matrices A and B, write down *defined* or *undefined* depending or whether the matrix product AB is defined or not. If it is defined, then compute the product AB.

(a)

$$A = \begin{pmatrix} 5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$$

(c)

$$A = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$$

(d)

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 2 & 2 & 3 \end{pmatrix},$$

(e)

$$A = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 \end{pmatrix},$$

Problem 4 (8 points)**.**

(1) Write down a matrix $A \in \mathbb{R}^{3\times 3}$ such that for all vectors $v \in \mathbb{R}^3$ we have

$$A \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 + v_3 \\ v_1 + 3v_2 \\ 2v_1 - 2v_2 + 2v_3 \end{pmatrix}$$

(2) Write down the definition of *row equivalence* from the lectures.

(3) Write the following 4×4 matrix B as a product of elementary matrices:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 4 & 1 \end{pmatrix}$$

(4) Write down the inverse of the above matrix B.

Problem 5 (10 points)**.**
Consider the following matrix $A \in \mathbb{R}^{3\times 3}$ and the vector $b \in \mathbb{R}^3$:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$$

(a) Bring the system to upper triangular form.
(b) Find $x \in \mathbb{R}^3$ such that $Ax = b$.
(c) Give a lower triangular matrix $L \in \mathbb{R}^{3\times 3}$ and an upper triangular matrix $U \in \mathbb{R}^{3\times 3}$ such that $A = LU$.

Problem 6 (6 points)**.**
Bring the following complex numbers into the form $z = a + b\mathbf{i}$ and write down their absolute value $|z|$:

(a)

$$z = \left(\mathbf{i}^3 - \mathbf{i}^5 + \mathbf{i}^7 + \mathbf{i}^{11}\right)\left(\mathbf{i}^2 + \mathbf{i}^4 + \mathbf{i}^6 + \mathbf{i}^7\right)$$

(b)

$$z = (3 + 5\mathbf{i})(1 + \mathbf{i})(4 - \mathbf{i})^2(3 - 5\mathbf{i})$$

(c)

$$z = \mathbf{i} + \mathbf{i}^2 + \mathbf{i}^3 + \mathbf{i}^4 + \cdots + \mathbf{i}^{99} + \mathbf{i}^{100}$$