Invertible Linear Mappings

A mapping $L : X \to Y$ is called **invertible** if there exists $L^{-1} : Y \to X$ such that

$$L^{-1} \circ L = \mathsf{Id}_X, \quad L \circ L^{-1} = \mathsf{Id}_Y \,.$$

We call L^{-1} the **inverse** of L.

Theorem

If $L : X \to Y$ is linear and invertible, then the inverse L^{-1} is linear.

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Proof

1. Let $\alpha \in \mathbb{R}$ and $y \in Y$. Set $x := L^{-1}(y)$. Since
$$L(\alpha x) = \alpha y,$$
we have
$$\alpha L^{-1}(y) = \alpha x = L^{-1}(\alpha y).$$

2. Let $y_1, y_2 \in Y$. Set $x_1 = L^{-1}(y_1)$ and $x_2 = L^{-1}(y_2)$. Since
$$L(x_1 + x_2) = y_1 + y_2$$
we have
$$L^{-1}(y_1 + y_2) = x_1 + x_2 = L^{-1}(y_1) + L^{-1}(y_2).$$

Hence L^{-1} is linear.

Theorem

Let $L : X \to Y$ *be an invertible linear mapping and let* $A \subseteq X$.

1. *A is linearly independent if and only if $L(A)$ is linearly independent.*
2. *We have* $\operatorname{span} L(A) = L(\operatorname{span} A)$.
3. *A is a basis if and only if $L(A)$ is a basis.*

Corollary

The image of a subspace under a linear mapping is again a subspace.

Corollary

Let $S \subseteq X$ *be a subspace of dimension* k *and let* $L : X \to Y$ *be an invertible linear mapping. Then* $L(S)$ *is a subspace of dimension* k.

Theorem

Let $v_1, \dots, v_n \in V$ *and let* $L : V \to W$ *be invertible. Write*

$$w_1 = Lv_1, \quad \dots, \quad w_n = Lv_n.$$

Then $v_1, \dots, v_n$ *is linearly dependent if and only if* $w_1, \dots, w_n$ *is linearly dependent .*

Proof.

If $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ with $0 = \alpha_1 v_1 + \cdots + \alpha_n v_n$, then

$$\begin{aligned} 0 = L(0) &= \alpha_1 L(v_1) + \cdots + \alpha_n L(v_n) \\ &= \alpha_1 w_1 + \cdots + \alpha_n w_n. \end{aligned}$$

So linear dependence of $v_1, \dots, v_n$ implies linear dependence of $w_1, \dots, w_n$.

Since L is invertible and linear, the inverse L^{-1} is linear. Hence the converse implication follows analogously. □

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$$w_1 = Lv_1, \quad \dots, \quad w_n = Lv_n.$$

Then $v_1, \dots, v_n$ *is a spanning set for* V *if and only if* $w_1, \dots, w_n$ *is a spanning set for* W *.*

Proof.

Let $w \in W$. There exists $v \in V$ with $L(v) = w$. There exist $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ with $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$. Consequently,

$$\begin{aligned} w = L(v) &= \alpha_1 L(v_1) + \cdots + \alpha_n L(v_n) \\ &= \alpha_1 w_1 + \cdots + \alpha_n w_n. \end{aligned}$$

So $v_1, \dots, v_n$ being a spanning set implies $w_1, \dots, w_n$ being a spanning set.

Since L is invertible and linear, the inverse L^{-1} is linear. Hence the converse implication follows analogously. □

Basis Transformations

Let V be an n-dimensional vector space with two different bases:

$$v_1, \ldots, v_n,$$

$$w_1, \ldots, w_n.$$

We let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be defined by

$$v_i = a_{i1} w_1 + \cdots + a_{in} w_n.$$

If $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{R}$ with

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = \beta_1 w_1 + \cdots + \beta_n w_n,$$

then

$$\begin{pmatrix} a_{11} & \ldots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

Indeed,

$$\sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} a_{ij} w_j = \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{n} a_{ij} \alpha_i \right)}_{=\beta_j} w_j$$

We call the matrix A the **basis transition matrix** from the basis $v_1, \ldots, v_n$ to the basis $w_1, \ldots, w_n$.

The basis transition matrix is necessarily invertible. Otherwise we had a linear dependence between basis vectors.

The inverse of a basis transition matrix is again a basis transition matrix, with the roles of the bases reversed.

If we have three bases

$$u_1, \ldots, u_n,$$

$$v_1, \ldots, v_n,$$

$$w_1, \ldots, w_n,$$

and let $B_{uv}, B_{vw} \in \mathbb{R}^{n \times n}$ denote the basis transition matrices from $u_1, \ldots, u_n$ to $v_1, \ldots, v_n$ and from $w_1, \ldots, w_n$ to $w_1, \ldots, w_n$, respectively, then

$$B_{uw} = B_{vw} \circ B_{uv} \in \mathbb{R}^{n \times n}$$

is the basis transition matrix from $u_1, \ldots, u_n$ to $w_1, \ldots, w_n$.

Every invertible matrix can be thought of as basis transition matrix with respect to some bases:

Pick $A \in \mathbb{R}^{n \times n}$ invertible and a basis $v_1, \ldots, v_n$. Then

$$w_1 = Av_1, \ldots, w_n = Av_n,$$

is a basis, and A^{-1} is the basis transition matrix from the basis $v_1, \ldots, v_n$ to the basis $w_1, \ldots, w_n$.

If $w_1, \ldots, w_n$ were linearly dependent with some coefficients $\beta_1, \ldots, \beta_n$, then an application of the inverse matrix A^{-1} would give coefficients $\alpha_1, \ldots, \alpha_n$ that yield a linear dependence of $v_1, \ldots, v_n$.

Questions?