Bases and Dimension

A vector space V has dimension $n \in \mathbb{N}_0$ if there exists a basis consisting of n different vectors of V. Then we call the vector space n-**dimensional**.

A vector space with finite dimension is called **finite dimensional**. A vector space that is not finite dimensional is called **infinite dimensional**.

We will gather some tools about finite dimensional vector spaces.

Examples

1. $\mathbb{R}^n$ has dimension n
2. $\mathbb{R}^{m \times n}$ has dimension $m \cdot n$.
3. The vector space of all polynomials over the real line is infinite-dimensional, but the vector space of all polynomials of maximum degree m has dimension $m + 1$. A basis is

$$1, x, x^2, \ldots, x^m.$$

4. The trivial vector space has dimension 0.

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Theorem

Let $v_1, \ldots, v_n$ be a basis of V. Let $w_1, \ldots, w_m \in V$ with $m > n$. Then $w_1, \ldots, w_m$ are linearly dependent.

Proof

Since $v_1, \ldots, v_n$ is a basis, there exists $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ with

$$w_i = a_{i1} v_1 + \cdots + a_{in} v_n.$$

Let $c_1, \ldots, c_m \in \mathbb{R}$. We have

$$w := c_1 w_1 + \cdots + c_m w_m = \sum_{i=1}^{m} \sum_{j=1}^{n} c_i a_{ij} v_j = \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{m} c_i a_{ij} \right)}_{=:\alpha_j} v_j.$$

Then w is zero if and only if each α_j is zero. But we have n homogeneous equations in $m > n$ variables

$$0 = c_1 a_{1j} + \cdots + c_m a_{mj},$$

so a non-trivial solution $c_1, \ldots, c_m \in \mathbb{R}$ exists.

Corollary

Let $v_1, \ldots, v_n$ *and* $w_1, \ldots, w_m$ *be two bases of* V*. Then* $n = m$*.*

In other words, the dimension of a finite-dimensional vector space is unique.

Corollary

Let $m < n$*. Let* $v_1, \ldots, v_m$ *be linearly independent members of an* n*-dimensional vector space. Then* $v_1, \ldots, v_m$ *is not a basis of* V*.*

Lemma

Let $v_1, \ldots, v_n$ *be linearly independent members of an* n*-dimensional vector space* V*. Then* $v_1, \ldots, v_n$ *is a basis of* V*.*

Proof

Let $w \in V$. Since V is n-dimensional, the set $w, v_1, \ldots, v_n$ is linearly dependent. There exist $\alpha_0, \alpha_1, \ldots, \alpha_n$, not all zero, with

$$0 = \alpha_0 w + \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

If α_0 were zero, then $v_1, \ldots, v_n$ would be linearly dependent. Hence $\alpha_0 \neq 0$. Thus

$$w = -\frac{\alpha_1}{\alpha_0} v_1 - \cdots - \frac{\alpha_n}{\alpha_0} v_n.$$

Hence w is in the span of $v_1, \ldots, v_n$. Since the $v_1, \ldots, v_n$ are linearly independent, the coefficients in the linear combination are unique. Hence $v_1, \ldots, v_n$ is a basis.

Lemma

Let $v_1, \ldots, v_n$ be a spanning set an n-dimensional vector space V. Then $v_1, \ldots, v_n$ is a basis of V.

Proof

Assume that $v_1, \ldots, v_n$ were linearly dependent. Then one of these vectors, say, v_i, would be a linear combination of the other vectors, and hence the reduced set $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$. would be spanning set for V.

Repeating this procedure until getting linearly independent spanning vectors $v_1, \ldots, v_k$ would prove that V has dimension $k < n$. But this is a contradiction.

Lemma

Let $r < n$. Let $v_1, \ldots, v_r$ be linearly independent members of an n-dimensional vector space V. Then there exist $v_{r+1}, \ldots, v_n \in V$ such that $v_1, \ldots, v_n$ is a basis of V.

Proof

Let $V_r = \text{span}\{v_1, \ldots, v_r\}$. We have $V \neq V_r$ since $r < n$. There exists $v_{r+1} \in V$ with $v_{r+1} \notin V_r$. Then $v_1, \ldots, v_r, v_{r+1}$ is linearly independent.

Repeat this procedure until n linearly independent vectors are found. We then have a basis.

Lemma

Let $r > n$. Let $v_1, \ldots, v_r$ be spanning vectors of an n-dimensional vector space V. Then there exists a subset of these spanning vectors that form a basis of V.

Proof

Since $r > n$, the vectors $v_1, \ldots, v_r$ are linearly dependent. Consequently, there exists v_i among them that is a linear combination of the others. Hence $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r$ is a set of $r - 1$ spanning vectors for V.

Repeat this procedure until $r = n$. We then have a basis.

Print and Frame

$$\text{Linearly Independent} \subseteq \text{Basis} \subseteq \text{Spanning Set}$$

Questions?