Eigenvalues and Eigenvectors

Let $A \in \mathbb{R}^{n \times n}$ be a matrix.

If $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n$, $v \neq 0$, with

$$Av = \lambda v,$$

then we call

1. λ an **eigenvalue** of A,
2. v an **eigenvector** of A,
3. and (λ, v) an **eigenpair** of A

Eigen, adjective: "own", "intrinsic".
First use in Linear Algebra in 1904 by David Hilbert.

Let $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n$ with $v \neq 0$.

The following are equivalent:

1. (λ, v) is an eigenpair of A
2. $Av = \lambda v$
3. $(A - \lambda \operatorname{Id}) v = 0$
4. $v \in \ker (A - \lambda \operatorname{Id})$

Conclusion: λ is an eigenvalue of A if $A - \lambda \operatorname{Id}$ is a singular matrix. This is the case exactly then if $\det (A - \lambda \operatorname{Id}) = 0$.

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This is the case exactly then if $\det(A - \lambda \operatorname{Id}) = 0$.

Let $A \in \mathbb{R}^{n \times n}$ be a matrix. The characteristic polynomial of A is

$$p_A(\lambda) := \det(A - \lambda \operatorname{Id}) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{pmatrix}$$

For $\lambda \in \mathbb{R}$ we have

$$p_A(\lambda) = 0 \quad \Longleftrightarrow \quad \det(A - \lambda \operatorname{Id}) = 0.$$

The matrix $A - \lambda \operatorname{Id}$ is singular if and only if λ is a root of the characteristic polynomial of A.

Let $A \in \mathbb{R}^{n \times n}$ and let p_A be the characteristic polynomial. By the Fundamental Theorem of Algebra, we can write

$$p_A(\lambda) = (\lambda - \lambda_1) \cdot \dots \cdot (\lambda - \lambda_n)$$

where the $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are the roots of the polynomial.

(The leading term λ^n has coefficient $(-1)^n$.)

The $\lambda_1, \dots, \lambda_n$ are not necessarily distinct. The **algebraic multiplicity** $\mu^{\mathrm{a}}(A, \lambda)$ is the number how often an eigenvalue appears as a root of the characteristic polynomial.

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Generally, the roots of a characteristic polynomial may be complex numbers. (Fundamental Theorem of Algebra)

Let $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{C}$, and $v \in \mathbb{C}^n$, $v \neq 0$.

We call λ an **eigenvalue** of A, we call v an **eigenvector** of A, and (λ, v) an **eigenpair** of A if

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Let $A \in \mathbb{C}^{n \times n}$, $\lambda \in \mathbb{C}$, and $v \in \mathbb{C}^{n}$, $v \neq 0$.

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The following are equivalent:

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3. $(A - \lambda \, \mathrm{Id}) \, v = 0$
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Example

$$A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}, \quad p_A(\lambda) = \det \begin{pmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{pmatrix},$$

We compute

$$p_A(\lambda) = -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda-1)^2$$

The roots of the polynomial p_A are precisely 0 and 1. The eigenvalue 0 has algebraic multiplicity 1 and the eigenvalue 1 has algebraic multiplicity 2:

$$\mu^{\mathrm{a}}(A,0) = 1, \quad \mu^{\mathrm{a}}(A,1) = 2,$$

Example

What are the eigenvectors?

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

Example

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad p_A(\lambda) = \det \begin{pmatrix} \cos(\theta) - \lambda & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - \lambda \end{pmatrix},$$

We compute

$$\begin{aligned} p_A(\lambda) &= \sin(\theta)^2 + (\cos(\theta) - \lambda)^2 \\ &= \sin(\theta)^2 + \cos(\theta)^2 - 2\lambda\cos(\theta) + \lambda^2 \\ &= \lambda^2 - 2\lambda\cos(\theta) + 1 \end{aligned}$$

Any root of this polynomial must satisfy

$$\cos(\theta)^2 - 1 = (\lambda - \cos(\theta))^2$$

The left-hand side is negative unless θ is an integer multiple of π, so the eigenvalues are complex unless θ is an integer multiple of π.

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Any root of this polynomial must satisfy

$$-\sin(\theta)^2 = (\lambda - \cos(\theta))^2$$

The left-hand side is negative unless θ is an integer multiple of π, so the eigenvalues are complex unless θ is an integer multiple of π.

Example

The eigenvalues are

$$\lambda_1 = \cos(\theta) + \sin(\theta)\mathbf{i}, \quad \lambda_2 = \cos(\theta) - \sin(\theta)\mathbf{i}.$$

We check that

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix},$$

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 \\ -\mathbf{i} \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ -\mathbf{i} \end{pmatrix}.$$

The characteristic polynomial p_A of $A \in \mathbb{C}^{n\times n}$ is defined as

$$p_A(\lambda) := \det\left(A - \lambda\,\mathrm{Id}\right) = \det\begin{pmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{pmatrix}$$

The scalar $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if it is a root of the characteristic polynomial.

Can we use special structures of the matrix to find the eigenvalues?

Example

Let $A \in \mathbb{C}^{n \times n}$ be a triangular matrix.

$$p_A(\lambda) = \det(A - \lambda \, \mathrm{Id}) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ 0 & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} - \lambda \end{pmatrix}$$

$$= (a_{11} - \lambda) \cdot (a_{22} - \lambda) \cdot \dots \cdot (a_{nn} - \lambda)$$

The eigenvalues of a triangular matrix are the diagonal elements:

$$p_A(\lambda) = \prod_{1 \le i \le n} (a_{ii} - \lambda) \, .$$

How to find the eigenvectors? If $\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{R}^n$, then the eigenvectors for that eigenvalue are the solutions of the homogeneous linear system of equations

$$(A - \lambda \operatorname{Id}) \cdot v = 0.$$

Possible strategy: Bring $A - \lambda \operatorname{Id}$ into reduced row echelon form and determine the nullspace from there.

Example

$$\ker\begin{pmatrix}1 & 2 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 3 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} = \left\{ x \in \mathbb{R}^6 \;\middle|\; \begin{array}{l} x_1 + 2x_2 = 0 \\ x_3 + 3x_4 - x_5 = 0 \\ x_6 = 0 \end{array} \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix}2\\-1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\3\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\\-3\\0\end{pmatrix} \right\}$$

Theorem

A matrix $A \in \mathbb{C}^{n\times n}$ is nonsingular if and only if 0 is not an eigenvalue of A.

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Proof.

The following are equivalent:

1. 0 is an eigenvalue of A
2. The matrix $A - 0\,\mathrm{Id}$ has a non-trivial kernel.
3. The matrix A has a non-trivial kernel.
4. There exists $v \in \mathbb{C}^n$, $v \neq 0$, with $Av = 0$.
5. The matrix A is singular.

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Theorem

Let $A \in \mathbb{C}^{n \times n}$*. Then*

$$\det(A) = \prod_{1 \leq i \leq n} \lambda_i$$

where $\lambda_1, \ldots, \lambda_n$ *are the eigenvalues of* A *(repeated according to algebraic multiplicity).*

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Let $A \in \mathbb{C}^{n \times n}$. Then

$$\det(A) = \prod_{1 \le i \le n} \lambda_i$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A (repeated according to algebraic multiplicity).

Proof.

We have

$$p_A(\lambda) = (-1)^n (\lambda - \lambda_1) \cdot \cdots \cdot (\lambda - \lambda_n)$$

Then $p_A(0) = \lambda_1 \cdot \cdots \cdot \lambda_n$. But we also have

$$p_A(0) = \det A$$

Hence the claim follows. □

Theorem

Let $A, B, S \in \mathbb{C}^{n\times n}$ *with* S *invertible and* $A = S^{-1}BS$. *Then*

$$p_A(\lambda) = p_B(\lambda).$$

Proof.

We have

$$\begin{aligned}
\det(A - \lambda\,\mathrm{Id}) &= \det\left(S^{-1}BS - \lambda\,\mathrm{Id}\right) \\
&= \det\left(S^{-1}BS - \lambda S^{-1}\,\mathrm{Id}\,S\right) \\
&= \det\left(S^{-1}(B - \lambda\,\mathrm{Id})\,S\right) \\
&= \det\left(S^{-1}\right)\det(B - \lambda\,\mathrm{Id})\det(S) \\
&= \det\left(S^{-1}\right)\det(S)\det(B - \lambda\,\mathrm{Id})
\end{aligned}$$

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Important Slide

Let $A \in \mathbb{C}^{n\times n}$. Then the following are equivalent:

1. A is invertible.
2. $Ax = 0$ if and only if $x = 0$.
3. $Ax = b$ always has a solution.
4. $\det(A) \neq 0$.
5. 0 is not an eigenvalue of A.
6. The row echelon form of A has only non-zero diagonal entries.
7. A has an n-dimensional row space.
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Let $A \in \mathbb{C}^{n \times n}$. Then the following are equivalent:

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Theorem

Let $A \in \mathbb{C}^{n\times n}$ with **different** *eigenvalues $\lambda_1, \ldots, \lambda_m$, $m \leq n$. Let $v_1, \ldots, v_m$ be respective eigenvectors for these eigenvalues. Then $v_1, \ldots, v_m$ are linearly independent.*

Proof.

Proof by induction. The claim is obviously true for $m = 1$.

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$$0 = \lambda_1 \cdot 0 = \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_1 v_2 + \cdots + \alpha_n \lambda_1 v_n,$$

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$$0 = A \cdot 0 = \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \cdots + \alpha_n \lambda_n v_n.$$

Substracting these equations from each other, we get

$$0 = \alpha_2 (\lambda_1 - \lambda_2) v_2 + \cdots + \alpha_n (\lambda_1 - \lambda_n) v_n, .$$

But then $\alpha_2 = \cdots = \alpha_n = 0$, and hence $\alpha_1 = 0$, contrary to our assumption. Hence $v_1, \ldots, v_m$ are linearly independent. $\square$

Questions?