Linear Combinations

Let V be a vector space.

Let $v_1, \ldots, v_n \in V$ be members of that vector space and let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ be scalars.

Then we call the sum

$$\alpha_1 v_1 + \cdots + \alpha_n v_n \in V$$

a **linear combination** of the vectors $v_1, \ldots, v_n$.

The vectors $v_1, \ldots, v_n \in V$ are called **linearly independent** if for all $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with

$$0 = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

we already have

$$\alpha_1 = \cdots = \alpha_n = 0.$$

General Case
A set of vectors $A \subseteq V$ is called *linearly independent* if every finite number of vectors $v_1, \ldots, v_n \in A$ is linearly independent. Otherwise A is called **linearly dependent**.

Remarks

1. This is a property of the entire set as a whole, not just single vectors.
2. If the coefficients are zero, then the linear combination is zero. If the linear combination is zero *and the vectors are linearly independent*, then the coefficients are zero.
3. Any set of vectors that includes the zero vector is **not linearly independent**.
4. What does it mean that a set is not linearly independent?
5. If $A \subseteq V$ is linearly independent and $\emptyset \subseteq A' \subseteq A$, then A' is linearly independent. Subsets of linearly independent sets are linearly independent.

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The **span** of the vectors $v_1, \ldots, v_n \in V$ is the set

$$\begin{aligned} S &= \operatorname{span}\{v_1, \ldots, v_n\} \\ &:= \{\alpha_1 v_1 + \cdots + \alpha_n v_n \mid \alpha_1, \ldots, \alpha_n \in \mathbb{R}\} \end{aligned}$$

We call the set $\{v_1, \ldots, v_n\} \subset V$ a **spanning set** of S.

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General Case

The **span** span$A$ of a set of vectors $A \subseteq V$ is the union of all spans of finite numbers of vectors $v_1, \dots, v_n \in A$. We call call A a **spanning set** of span$A$.

In other words, span$A$ is the set of all (finite) linear combinations of vectors taken from A.

Remarks

1. If A is empty, then span$A$ contains the zero vector. *The sum of no summands is zero.*
2. Every finite number of vectors $v_1, \ldots, v_m$ is the spanning set for some vector space, namely span$\{v_1, \ldots, v_m\}$.
3. Every set of vectors A is the spanning set for some vector space, namely span$A$.
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Proof

1. We have $0 \in S$ because

$$0 = 0v_1 + \cdots + 0v_n.$$

2. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. Then

$$\alpha\left(\alpha_1 v_1 + \cdots + \alpha_n v_n\right) = \alpha\alpha_1 v_1 + \cdots + \alpha\alpha_n v_n \in S.$$

3. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $\beta_1, \ldots, \beta_n \in \mathbb{R}$.

$$\begin{aligned}&(\alpha_1 v_1 + \cdots + \alpha_n v_n) + (\beta_1 v_1 + \cdots + \beta_n v_n) \\ &= (\alpha_1 + \beta_1)v_1 + \cdots + (\alpha_n + \beta_n)v_n \in S.\end{aligned}$$

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1. The span of any set contains the zero vector.
2. Scaling any (finite) linear combination of vectors from A gives a (finite) linear combination of vectors A.
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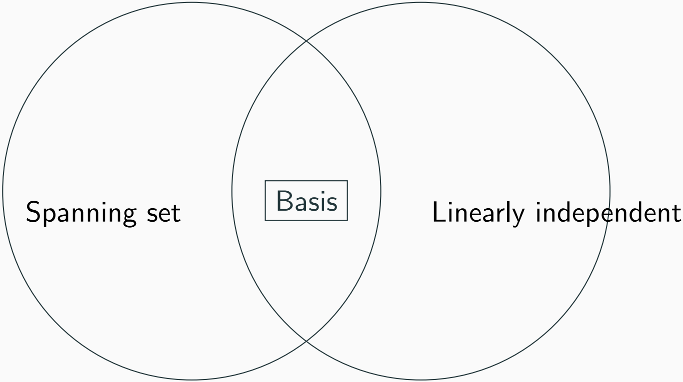

Example

Let $V = \mathbb{R}^3$. Then

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

is a spanning set of $\mathbb{R}^3$. A basis is given by

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

or by

$$\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Example

Let $V = \mathbb{R}^{2\times 2}$. Then a basis is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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Example

Let $V = \mathcal{P}^3$ the vector space of polynomials over $\mathbb{R}$ up to degree 3. Then a basis is given by

$$x^3, \qquad x^2, \qquad x, \qquad 1$$

or by

$$x^3 + x, \qquad x^2 + 1, \qquad x^2 + x, \qquad x^3 + 1.$$

A proper subspace is spanned by

$$x^2, \qquad x.$$

The polynomials

$$x, \qquad 1, \qquad x + 1$$

is linearly dependent.

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If $v_1, \ldots, v_n$ is a basis of V, then every vector $v \in V$ can be written as linear combination of these vectors in a unique manner.

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Proof

1. Since the vectors $v_1, \ldots, v_n$ are a basis of V, they are also a spanning set of V. Hence for every $v \in V$ there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that
$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$
2. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $\beta_1, \ldots, \beta_n \in \mathbb{R}$ such that
$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n = \beta_1 v_1 + \cdots + \beta_n v_n.$$
Then
$$0 = (\alpha_1 - \beta_1) v_1 + \cdots + (\alpha_n - \beta_n) v_n.$$
Hence $\alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n$. Uniqueness!

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Example

Find $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\alpha_1 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ 4 \end{pmatrix}$$

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Solution:

$$\alpha_1 = 3, \quad \alpha_2 = 2, \quad \alpha_3 = -4,$$

The moral of the story: Writing a vector in $\mathbb{R}^n$ as a linear combination of basis vectors is equivalent to solving a linear system of equations.

Many sides of the same coin:

Computing a matrix–vector product Ax

$\iff$

Taking a linear combination of the column vectors of A with the coefficients $x_1, \ldots, x_n$.

Solving a linear system of equations $Ax = b$

$\iff$

Expressing a vector in terms of the columns of A, with the coefficients $x_1, \ldots, x_n$.

Many sides of the same coin:

1. Linear systems of equations
2. Matrix–vector multiplications
3. Linear combinations of vectors
4. Geometric transformations

Theorem

If $v_1, \ldots, v_n$ is a spanning set of V that is not a basis, then every vector $v \in V$ can be written as linear combination of $v_1, \ldots, v_n$ in a non-unique manner.

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Proof

Let $v \in \operatorname{span}\{v_1, \ldots, v_n\}$. Then there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

Since $v_1, \ldots, v_n$ is not a basis, there exist $\beta_1, \ldots, \beta_n \in \mathbb{R}$, not all zero, satisfying

$$0 = \beta_1 v_1 + \cdots + \beta_n v_n.$$

For every $\gamma \in \mathbb{R}$ we have

$$v = (\alpha_1 + \gamma\beta_1) v_1 + \cdots + (\alpha_n + \gamma\beta_n) v_n.$$

Re-enter the elementary row operations.

Theorem

Let $v_1, \dots, v_m \in V$ be vectors and let $w_1, \dots, w_m$ be obtained by one of the following three procedures:

- *Switching vectors:*

$$w_1 = v_1, \quad \dots, \quad w_{i-1} = v_{i-1}, w_i = v_j, w_{i+1} = v_{i+1},$$
$$\dots, \quad w_{j-1} = v_{j-1}, w_j = v_i, w_{j+1} = v_{j+1}, \quad \dots, \quad w_n = v_n$$

- *Scaling the i-th vector by a nonzero factor $\alpha \in \mathbb{R}$:*

$$w_1 = v_1, \dots, w_{i-1} = v_{i-1}, w_i = \alpha v_i, w_{i+1} = v_{i+1}, \dots, w_n = v_n.$$

- *Adding the α-th multiple of v_j to the i-th vector:*

$$w_1 = v_1, \dots, w_{j-1} = v_{j-1}, w_j = v_j + \alpha v_j, w_{j+1} = v_{j+1}, \dots w_n = v_n.$$

For each of these three constructions, $w_1, \dots, w_m$ has the same span as $v_1, \dots, v_m$, and $w_1, \dots, w_m$ is linearly independent if and only if $v_1, \dots, v_m$ is.

More Examples:

1. The vector space $\mathcal{P}_d$ of polynomials with maximal degree d has a basis:

$$A = \left\{1, x, x^2, x^3, \ldots, x^d\right\}.$$

2. The vector space $\mathcal{P}$ of polynomials has a basis:

$$A = \left\{1, x, x^2, x^3, \ldots, x^k, x^{k+1}, \ldots\right\}.$$

3. Let $F(\mathbb{R}, \mathbb{R})$ be the vector of functions $f : \mathbb{R} \to \mathbb{R}$ from the real numbers into the real numbers.
   Does it have a basis?

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Answer: You Decide!

Technically, a basis exists for common vector spaces like $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$. Whether **every** vector space has a basis, however, is equivalent to an axiom of set theory (Axiom of Choice).

The vast majority of mathematicians assume the axiom of choice to be true, while some prefer to work without it.

Questions?