General Information

1. A small Matrix Workout is available for download on the homepage.
2. Prospective office hours next week: Monday, Wednesday, 1:30 - 3:00 pm.
3. You are welcome to ask questions if you need help improving.

Linear Mappings and Matrices

Every matrix $A \in \mathbb{R}^{m\times n}$ gives rise to a linear mapping via matrix-vector multiplication.

$$L_A : \mathbb{R}^n \to \mathbb{R}^m, \quad x \mapsto A \cdot x$$

We now show how to represent every linear mapping

$$L : \mathbb{R}^n \to \mathbb{R}^m$$

by a matrix.

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ and let $e_1, \ldots, e_m$ be the standard basis of $\mathbb{R}^m$. Suppose that

$$L(e_i) = a_{1i}e_1 + \cdots + a_{mi}e_m.$$

Then the matrix

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$$

represents the linear mapping L. We have $L = L_A$.

Let $A = \{a_1, \ldots, a_n\}$ be a basis of a vector space V. This basis can be identified with an *invertible* linear mapping

$$I_A : \mathbb{R}^n \to V, \quad (\alpha_i)_{1 \le i \le n} \mapsto \sum_{1 \le i \le n} \alpha_i a_i.$$

This mapping is fully described by the values it assumes at the unit vectors $e_1, \ldots, e_n$:

$$I_A(e_1) = a_1, \quad \ldots \quad I_A(e_n) = a_n.$$

Assume we have got two bases of the vector space V,

$$A = \{a_1, \ldots, a_n\}, \quad B = \{b_1, \ldots, b_n\},$$

and the associated linear mappings

$$I_A : \mathbb{R}^n \to V, \quad I_B : \mathbb{R}^n \to V.$$

The basis transition $\mathcal{B}_{ab} : \mathbb{R}^n \to \mathbb{R}^n$ then satisfies

$$I_B \circ \mathcal{B}_{ab} = I_A.$$

In fact,

$$\mathcal{B}_{ab} = I_B^{-1} I_A.$$

The choice of bases

$$A = \{a_1, \ldots, a_n\} \subseteq V, \quad B = \{b_1, \ldots, b_m\} \subseteq W,$$

corresponds to the choice of invertible linear mappings

$$I_A : \mathbb{R}^n \to V, \quad I_B : \mathbb{R}^m \to W.$$

With respect to these choices of bases, every linear mapping $T : V \to W$ can be represented by a matrix $M \in \mathbb{R}^{m \times n}$ as follows:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ I_A^{-1} \big\downarrow & & I_B \big\uparrow \\ \mathbb{R}^n & \xrightarrow{L_M} & \mathbb{R}^m \end{array}$$

Two matrices $A, B \in \mathbb{R}^{n \times n}$ are called *similar* if there exists an invertible mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$A = T^{-1} \cdot B \cdot T.$$

Basic intuition: Similar matrices describe the same linear mapping but with regards to different bases.

Similarlity is an equivalence relation:

1. If $A \in \mathbb{R}^{n\times n}$ is similar to $B \in \mathbb{R}^{n\times n}$, then B is similar to A.

$$A = T^{-1}BT \quad \Longrightarrow \quad B = TAT^{-1}.$$

2. Every matrix is similar to itself.

$$A = \mathrm{Id}_n \, A \, \mathrm{Id}_n \, .$$

3. If $A \in \mathbb{R}^{n\times n}$ is similar to $B \in \mathbb{R}^{n\times n}$ and B is similar to $C \in \mathbb{R}^{n\times n}$, then A is similar to C.

$$A = T^{-1}BT, \quad B = S^{-1}CS \quad \Longrightarrow \quad A = T^{-1}S^{-1}CST.$$

Similar matrices $A, B \in \mathbb{R}^{n\times n}$ have many things in common.

1. Regular or Singular.

$$A^{-1} = \left(T^{-1}BT\right)^{-1} = T^{-1}B^{-1}T.$$

2. Determinant.

$$\det(A) = \det(T^{-1}BT) = \det(T^{-1})\det(B)\det(T) = \det(B).$$

3. Rank.

Questions?