Subspaces for a Linear Mapping

Let $L : V \to W$ be a linear mapping.

The **kernel** of L is the subspace

$$\ker L := \{ v \in V \mid Lv = 0 \} .$$

The **range** of L is the subspace

$$\begin{aligned} \operatorname{rng} L &:= \{ w \in W \mid \text{ There exists } v \in V : w = Lv \} \\ &:= \{ L(v) \in W \mid v \in V \} . \end{aligned}$$

Theorem

$\ker L$ *is a subspace of* V.

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Proof

1. We have $L(0) = 0 \in W$, so $0 \in V$ is contained in $\ker L$.
2. Let $v \in \ker L$ and $\alpha \in \mathbb{R}$. Then

$$0 = \alpha 0 = \alpha L(v) = L(\alpha v).$$

   Hence $\alpha v \in \ker L$.

3. Let $v_1, v_2 \in \ker L$. Then

$$0 = 0 + 0 = L(v_1) + L(v_2) = L(v_1 + v_2).$$

   Hence $v_1 + v_2 \in \ker L$.

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Theorem

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Proof

1. We have $L(0) = 0$. Since $0 \in V$, we have $0 \in \operatorname{rng} L$.
2. Let $w \in \operatorname{rng} L$ and $\alpha \in \mathbb{R}$. There exists $v \in V$ with $L(v) = w$. Then

$$\alpha w = \alpha L(v) = L(\alpha v).$$

   Hence $\alpha w \in \operatorname{rng} L$.
3. Let $w_1, w_2 \in \operatorname{rng} L$. There exist $v_1, v_2 \in V$ with

$$w_1 = L(v_1), \quad w_2 = L(v_2).$$

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Questions?