# Math 102 – Fall Quarter 2017 – Midterm I

Full name: ______________________________

Student ID: ______________________________

**Instructions:**

(1) Please print your full name and your student ID.

(2) Using cheatsheets, calculators, books, or phones is **not** allowed.

(3) You have 50 minutes to complete the test.

| Problem | Points |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| $\Sigma$ | |

**Problem 1** (10 points)**.**
To each statement, write TRUE or FALSE, depending on whether the statement is true or false.

(1) The $n \times n$ identity matrix $\mathrm{Id}_n$ has all entries 1. F

(2) The following matrix–matrix product is computed correctly: F

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(3) If $A \in \mathbb{R}^{2\times 2}$ with $A^2 = 0$, then $A = 0$. F

(4) The following matrix $A \in \mathbb{R}^{2\times 2}$ is invertible for all values $\alpha \in \mathbb{R}$: F

$$A = \begin{pmatrix} 5 & 0 \\ 0 & \alpha \end{pmatrix}.$$

(5) The following matrix $B \in \mathbb{R}^{2\times 2}$ is invertible for all values $\beta \in \mathbb{R}$: T

$$B = \begin{pmatrix} 5 & \beta \\ 0 & 5 \end{pmatrix}.$$

(6) For all matrices $A, B \in \mathbb{R}^{n\times n}$ we have $AB = BA$. F

(7) For invertible matrices $A, B \in \mathbb{R}^{n\times n}$ we have $AB = BA$. F

(8) For all invertible matrices $A, B \in \mathbb{R}^{n\times n}$ the product $AB$ is invertible with $(AB)^{-1} = A^{-1}B^{-1}$. F

(9) The following matrix–matrix product is computed correctly: F

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}.$$

(10) The following matrix–matrix product is computed correctly: T

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -5 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & 5 \end{pmatrix}.$$

**Problem 2** (8 points)**.**
For each statement, write TRUE or FALSE, depending on whether the statement is true or false.
**If it is false, then write down a counterexample. No proof is necessary.**

(1) For every field $\mathbb{F}$ and every $y \in \mathbb{F}$ there exists $x \in \mathbb{F}$ with $x^2 = y$.

F For instance, take $\mathbb{F} = \mathbb{Q}$ and $y = 2$. Then there is no $x \in \mathbb{Q}$ such that $x^2 = y$.

(2) For every field $\mathbb{F}$ and all $a, b, c \in \mathbb{F}$ we have $a(b + c) = ab + ac$.

T

(3) For every field $\mathbb{F}$ and all $a, b, c \in \mathbb{F}$ we have $(a + b)c = ac + bc$.

T

(4) For all $A, B \in \mathbb{R}^{3\times 3}$ we have $A + B = B + A$.

T

(5) If $A, B \in \mathbb{R}^{2\times 2}$ are upper triangular, then their product $AB$ is upper triangular.

T

(6) All diagonal matrices are invertible.

F For instance $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a diagonal matrix which is not invertible.

(7) Every real number $a \in \mathbb{R}$ can be represented as $a = p/q$ with $p, q \in \mathbb{Z}$ and $q \neq 0$.

F For instance $\sqrt{2} \in \mathbb{R}$ but $\sqrt{2}$ is not a rational number, so it cannot be written as $p/q$ with $p, q \in \mathbb{Z}$.

(8) If $A \in \mathbb{R}^{2\times 2}$ is invertible, then $AA^{-1}A = A$.

T

**Problem 3** (10 points)**.**
For each the following pairs of matrices $A$ and $B$, write down *defined* or *undefined* depending or whether the matrix product $AB$ is defined or not. If it is defined, then compute the product $AB$.

(a)

$$A = \begin{pmatrix} 5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Undefined.

(b)

$$A = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$$

Defined.

$$AB = \begin{pmatrix} 2 & 3 & 1 \\ -2 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

(c)

$$A = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$$

Defined.

$$AB = (-5 - 2 + 3) = (-4).$$

(d)

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 2 & 2 & 3 \end{pmatrix},$$

Defined.

$$AB = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & -3 & 0 \\ 15 & 6 & 6 & 9 \end{pmatrix}$$

(e)

$$A = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 \end{pmatrix},$$

Defined.

$$AB = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$

**Problem 4** (8 points)**.**

(1) Write down a matrix $A \in \mathbb{R}^{3\times 3}$ such that for all vectors $v \in \mathbb{R}^3$ we have

$$A \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 + v_3 \\ v_1 + 3v_2 \\ 2v_1 - 2v_2 + 2v_3 \end{pmatrix}$$

(2) Write down the definition of *row equivalence* from the lectures.

(3) Write the following $4 \times 4$ matrix $B$ as a product of elementary matrices:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 4 & 1 \end{pmatrix}$$

(4) Write down the inverse of the above matrix $B$.

**Solution.**

(1) $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & -2 & 2 \end{pmatrix}$

(2) An $m \times n$ matrix $A$ is row equivalent to an $m \times n$ matrix $B$ if there exist a set of $m \times m$ elementary matrices $E_1, \dots, E_k$ such that $A = E_1 E_2 \dots E_k B$.

(3) We perform the following row operations to turn $B$ into the identity:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 4 & 1 \end{pmatrix} \xrightarrow[\text{to R4}]{\text{add } -5\cdot\text{R1}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{pmatrix} \xrightarrow[\text{to R4}]{\text{add } -4\cdot\text{R3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{multiply R2 by } -1} I$$

If we translate the row operations into multiplications by elementary matrices, we see that

$$I = E_3 E_2 E_1 B$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{pmatrix} E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therfore we can write $B$ as a product of elementary matrices as

$$B = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{pmatrix}.$$

(4) From above, we see $I = (E_3 E_2 E_1) B$ which means

$$B^{-1} = E_3 E_2 E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & -4 & 1 \end{pmatrix}.$$

**Problem 5** (10 points)**.**
Consider the following matrix $A \in \mathbb{R}^{3\times 3}$ and the vector $b \in \mathbb{R}^3$:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$$

(a) Bring the system to upper triangular form.
(b) Find $x \in \mathbb{R}^3$ such that $Ax = b$.
(c) Give a lower triangular matrix $L \in \mathbb{R}^{3\times 3}$ and an upper triangular matrix $U \in \mathbb{R}^{3\times 3}$ such that $A = LU$.

(a) We perform the following row operations to bring the system to upper triangular form:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 1 & 1 & 1 & 3 \\ 3 & 3 & 1 & 4 \end{array}\right) \xrightarrow[\text{to R2}]{\text{add } -1\cdot \text{ R1}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & -1 & -2 & 5 \\ 3 & 3 & 1 & 4 \end{array}\right) \xrightarrow[\text{to R3}]{\text{add } -3\cdot \text{ R1}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & -1 & -2 & 5 \\ 0 & -3 & -8 & 10 \end{array}\right)$$

$$\xrightarrow[\text{to R3}]{\text{add } -3\cdot\text{R2}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & -1 & -2 & 5 \\ 0 & 0 & -2 & -5 \end{array}\right)$$

(b) The system in upper triangular form reads

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= -2 \\ -x_2 - 2x_3 &= 5 \\ -2x_3 &= -5. \end{aligned}$$

Solving this we find $x_3 = 5/2$, $x_2 = -(5 + 2 \cdot (5/2)) = -10$ and $x_1 = -2 - 2 \cdot (-10) - 3 \cdot (5/2) = 21/2$. Therefore the solution is

$$x = \begin{pmatrix} 21/2 \\ -10 \\ 5/2 \end{pmatrix}.$$

(c) Let

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -2 \end{pmatrix}.$$

Then our work in part (a) shows that $U$ is row equivalent to $A$ and that $U = E_3E_2E_1A$ where $E_1, E_2, E_3$ are the elementary matrices

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}.$$

Solving for $A$ in the equation above we have $A = (E_1^{-1}E_2^{-1}E_3^{-1})U$. Since each of $E_1, E_2, E_3$ are lower triangular, so is $E_1^{-1}E_2^{-1}E_3^{-1}$ so in the LU-decomposition we can take

$$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix}.$$

**Problem 6** (6 points)**.**
Bring the following complex numbers into the form $z = a + b\mathbf{i}$ and write down their absolute value $|z|$:

(a)

$$z = \left(\mathbf{i}^3 - \mathbf{i}^5 + \mathbf{i}^7 + \mathbf{i}^{11}\right)\left(\mathbf{i}^2 + \mathbf{i}^4 + \mathbf{i}^6 + \mathbf{i}^7\right)$$

(b)

$$z = (3 + 5\mathbf{i})(1 + \mathbf{i})(4 - \mathbf{i})^2(3 - 5\mathbf{i})$$

(c)

$$z = \mathbf{i} + \mathbf{i}^2 + \mathbf{i}^3 + \mathbf{i}^4 + \cdots + \mathbf{i}^{99} + \mathbf{i}^{100}$$

**Solution.**

(a)

$$z = (i^3 - i^5 + i^7 + i^1 1)(i^2 + i^4 + i^6 + i^7) = (-i - i - i - i)(-1 + 1 - 1 - i) = (-4i)(-1 - i) = -4 + 4i$$

Also $|z| = \sqrt{4^2 + 4^2} = \sqrt{32}$.

(b)

$$z = (3{+}5i)(1{+}i)(4{-}i)^2(3{-}5i) = (3{+}5i)(3{-}5i)(1{+}i)(-8i{+}15) = 34(1{+}i)(-8i{+}15) = 34(7i{+}23) = 782{+}238i.$$

Therefore $|z| = \sqrt{782^2 + 238^2}$.

(c)

$$z = i(1 + i + i^2 + \cdots + i^{99}) = i\frac{1 - i^{100}}{1 - i} = i\frac{1 - 1}{1 - i} = 0$$

where the second equality comes from the formula for a geometric series. Finally $|z| = |0| = 0$.

Alternatively, we can notice that

$$z = i + i^2 + i^3 + \cdots + i^{100} = (i - 1 - i + 1) + (i - 1 - i + 1) + \cdots + (i - 1 - i + 1) = 0.$$