HW 2

1) $$[A \,\vdots\, b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & \alpha & 3 \end{array}\right] \underset{-2R_1+R_3}{\overset{R_1+R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 6 & 4 & 3 \\ 0 & -6 & \alpha-2 & 1 \end{array}\right]$$

$$\overset{R_2+R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 6 & 4 & 3 \\ 0 & 0 & \alpha+2 & 4 \end{array}\right]$$

which has a unique soln $\iff \alpha+2 \neq 0 \iff \boxed{\alpha \neq -2}$

If so, i.e. if $\alpha \neq -2$

$$\Rightarrow \quad x_3 = \boxed{\frac{4}{\alpha+2}}$$

$$x_2 = \frac{3 - 4\left(\frac{4}{\alpha+2}\right)}{6} = \frac{\frac{3\alpha+6-48}{3(\alpha+2)}}{6} = \boxed{\frac{3\alpha-42}{18(\alpha+2)}}$$

$$x_1 = 3 - x_3 - 2x_2$$

$$= 3 - \frac{4}{\alpha+2} - \cancel{2}\left(\frac{3\alpha-42}{\cancel{18}_{9}(\alpha+2)}\right)$$

$$= \frac{3\cdot 9(\alpha+2) - 4\cdot 9 - (3\alpha-42)}{9(\alpha+2)} = \boxed{\frac{24\alpha+60}{9(\alpha+2)}}$$

2) a)

$$\underset{A}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} \underset{B}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A \neq 0 \qquad B \neq 0$$

b)

$$\underset{A}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} \underset{C}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix}} \underset{C}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but $A \neq C$.

c)

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \quad \underset{B}{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$$

non-singular, but $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ singular

d)

$$\underset{A}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{B}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Id_2$$

but

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq Id_3$$

3) $A_{n\times n}$, $k \in \mathbb{N}$ s.t. $A^{k+1} = 0$

Then $C_1 = (Id_n - A)(Id_n + A + \cdots + A^k)$

$= Id_n + \cancel{A} + \cancel{\cdots} + \cancel{A^k} - \cancel{A} - \cancel{A^2} - \cdots - \cancel{A^k} - A^{k+1}$

$= Id_n - \cancel{A^{k+1}}^{\,0}$

$= Id_n$

and similarly, $C_2 = (Id_n + \cdots + A^k)(Id_n - A)$

$= Id_n - A^{k+1}$

$= Id_n$

Therefore the matrix $\underset{n\times n}{B} = Id_n - A$ is invertible (non-Sing.)

since there exists a matrix $\underset{n\times n}{C} = Id_n + \cdots + A^k$

such that $BC = CB = Id_n$. ✓

4) $A = \begin{pmatrix} .5 & -.1 \\ .2 & .6 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The exact soln is : $\left(\begin{array}{cc|c} .5 & -.1 & 1 \\ .2 & .6 & 1 \end{array}\right) \underset{\sim}{-2/5 R_1 + R_2} \left(\begin{array}{cc|c} .5 & -.1 & 1 \\ 0 & .64 & .6 \end{array}\right)$

$\Rightarrow .64 \tilde{x}_2 = .6 \Rightarrow \tilde{x}_2 = \frac{.6}{.64} = \boxed{.9375}$

& $.5\tilde{x}_1 - (.1)\tilde{x}_2 = 1 \Rightarrow \tilde{x}_1 = \frac{1.09375}{.5} = \boxed{2.1875}$

Now let $\underline{x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$

$x_1 = x_0 + (b - Ax_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$x_2 = x_1 + (b - Ax_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} .4 \\ .8 \end{pmatrix} = \underline{\begin{pmatrix} 1.6 \\ 1.2 \end{pmatrix}}$

$x_3 = x_2 + (b - Ax_2) = \begin{pmatrix} 1.6 \\ 1.2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} .64 \\ 1.04 \end{pmatrix} = \underline{\begin{pmatrix} 1.96 \\ 1.16 \end{pmatrix}}$

$x_4 = x_3 + (b - Ax_3) = \begin{pmatrix} 1.96 \\ 1.16 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} .864 \\ 1.088 \end{pmatrix} = \underline{\begin{pmatrix} 2.096 \\ 1.072 \end{pmatrix}}$

Now we notece the first entries of x_1, x_2, x_3, x_4 are 1, 1.6, 1.96, 2.096, which is a monotonically increasing sequence whose rate of increase is getting smaller. So it seems plausible this sequence limits towards $2.1875 = \tilde{x}_1$

Similarly, the second entries, 1, 1.2, 1.16, 1.072 are monotonically decreasing after the ~~second~~ first term, & it is plausible that they tend towards $.9375 = \tilde{x}_2$.