Math 102 HW 3

Exercise 1.

(a) Suppose that A is row equivalent to B. Recall that an $m \times n$ matrix A is row equivalent to an $m \times n$ matrix B if there exists a sequence $E_1, E_2, \dots E_k$ of $m \times m$ elementary matrices such that

$$A = E_k E_{k-1} \dots E_1 B.$$

Now we have the following important fact: any elementary matrix E is invertible, and its inverse is also an elementary matrix (of the same type). You should check this fact by looking at the 3 types of elementary matrices. Using this, we can write

$$B = (E_1)^{-1} \dots E_{k-1}^{-1} E_k^{-1} A$$

(note how the order is reversed). By the fact, each E_i^{-1} is also an elementary matrix. Thus by the definition of row equivalence, B is row equivalent to A.

(b) Suppose A is row equivalent to B and B is row equivalent to C. This means that there are elementary matrices $E_1, \dots E_k$ and $F_1, \dots F_j$ such that

$$A = E_k E_{k-1} \dots E_1 B, \qquad B = F_j F_{j-1} \dots F_1 C.$$

Substituting the second equation into the first we get

$$A = E_k E_{k-1} \dots E_1 F_j F_{j-1} \dots F_1 C.$$

Since each of the E_s and F_r are elementary matrices, this shows that A is row equivalent to C.

Exercise 2.

(a) For the first two we just distribute:

$$(5 + 3i)(6 + 3i) = 30 + 15i + 18i - 9 = 21 + 33i.$$

(b)

$$(1 + i)(1 - i) = 1 + i - i + 1 = 2.$$

(c) First compute $(2 + i)^{-1} = \frac{1}{2+i} = \frac{1}{2+i}\left(\frac{2-i}{2-i}\right) = \frac{2-i}{5}$. Therefore

$$i(2 + i)^{-1} = i\frac{2 - i}{5} = \frac{1}{5} + \frac{2}{5}i.$$

(d)

$$(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}i) \cdot (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}i) = \frac{1}{2} + \frac{2}{\sqrt{2}\sqrt{3}}i - \frac{1}{3} = \frac{1}{6} - \frac{2}{\sqrt{6}}i.$$

Exercise 3.

Note: instead of "absolute value" we use the term "norm" for $|a+bi| = \sqrt{a^2+b^2}$.

(a)

We have $1/i = -i$, $\frac{1}{1+i} = \frac{1}{1+i}\left(\frac{1-i}{1-i}\right) = \frac{1-i}{2}$ and similarly $\frac{1}{1-i} = \frac{1+i}{2}$. Therefore

$$\frac{1}{i} + \frac{1}{1-i} + \frac{1}{1-i} = -i + 1/2(1-i) + 1/2(1+i) = 1-i$$

and its norm is $|1-i| = \sqrt{1^2+1^2} = \sqrt{2}$.

(b) We have $\frac{5-3i}{i} = (-i)(5-3i) = -3-5i$ and also $(1+2i)(1-2i) = (1+2^2) = 5$. Therefore

$$\frac{5-3i}{i} + (1+2i)(1-2i) = -3-5i+5 = 2-5i$$

and its norm is $|2-5i| = \sqrt{2^2+5^2} = \sqrt{29}$.

(c) Since $i^4 = 1$ we have $i^8 = (i^4)^2 = 1^2 = 1$, and its norm is $|1| = 1$.

(d) We compute

$$\begin{aligned}(1+\sqrt{2}i)(2-3i)(3+i) = (1+\sqrt{2}i)(6+2i-9i+3) &= (1+\sqrt{2}i)(9-7i) = 9-7i+9\sqrt{2}i+7\sqrt{2}i \\ &= (9+7\sqrt{2}) + (9\sqrt{2}-7)i.\end{aligned}$$

Its norm is $\sqrt{(9+7\sqrt{2})^2 + (9\sqrt{2}-7)^2} = \sqrt{390}$.

Exercise 4.

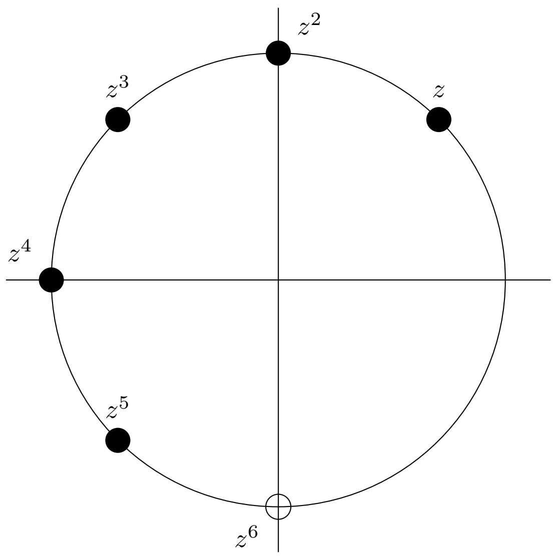
(a) We compute

$$z^2 = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = 1/2 + i/2 + i/2 - 1/2 = i.$$

Therefore

$$\begin{aligned} z^3 &= z^2 \cdot z = iz = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \\ z^4 &= (z^2)^2 = i^2 = -1 \\ z^5 &= (z^4) * z = -z = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i. \end{aligned}$$

(b) All these points belong to the unit circle, as indicated. The (correct) guess



for z^6 is $z^6 = -i$.

Exercise 4.

(a) Let $z_1 = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}i$ and $z_2 = \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$ so that

$$v = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

We must check that $|z_1|^2 + |z_2|^2 = 1$. First we compute

$$\begin{aligned} |z_1|^2 &= \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 \\ &= \frac{3}{8} + \frac{1}{8} = 1/2. \end{aligned}$$

Similarly we have

$$|z_2|^2 = \left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2 = \frac{1}{8} + \frac{3}{8} = 1/2.$$

Therefore $|z_1|^2 + |z_2|^2 = 1/2 + 1/2 = 1$ so v is a qubit.

(b) Using the definition of matrix-vector multiplication we have

$$\begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\alpha i - \beta) \\ \frac{1}{\sqrt{2}}(\alpha - i\beta) \end{pmatrix}$$

Therefore

$$\begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(-\beta + \alpha i) \\ \frac{1}{\sqrt{2}}(\alpha - \beta i) \end{pmatrix}.$$

(c) We assume that $|\alpha|^2 + |\beta|^2 = 1$ and we must check that $|\gamma|^2 + |\delta|^2 = 1$, where γ and δ are as above. We use the general formula $|z|^2 = z\overline{z}$ (works for any complex number) to compute

$$\begin{aligned}
|\gamma|^2 = \gamma\overline{\gamma} &= (\frac{1}{\sqrt{2}}(-\beta + \alpha i))\overline{\frac{1}{\sqrt{2}}(-\beta + \alpha i)} \\
&= 1/2(-\beta + \alpha i)\overline{-\beta + \alpha i} \\
&= \frac{1}{2}(-\beta + \alpha i)(-\overline{\beta} + \overline{\alpha} \cdot (-i)) \\
&= \frac{1}{2}(\beta\overline{\beta} + i\beta\overline{\alpha} - i\alpha\overline{\beta} + \alpha\overline{\alpha}) \\
&= \frac{1}{2}(|\beta|^2 + |\alpha|^2 + (\beta\overline{\alpha} - \alpha\overline{\beta})i) \\
&= \frac{1}{2}(1 + (\beta\overline{\alpha} - \alpha\overline{\beta})i).
\end{aligned}$$

We repeatedly used the facts that for complex numbers z, w we have $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z} \cdot \overline{w}$. We similarly computes that

$$|\delta|^2 = \frac{1}{2}(1 + (\alpha\overline{\beta} - \beta\overline{\alpha})i)$$

and now we see that $|\gamma|^2 + |\delta^2| = 1$.