Math 102 HW 4

Exercise 1.

(a) To reduce fractions of complex numbers, multiply both the top and the bottom by the conjugate of the bottom:

$$\begin{aligned} z_1 &= \frac{2i}{3+4i} + \frac{i}{3-4i} \\ &= \frac{2i}{3+4i}\left(\frac{\overline{3+4i}}{\overline{3+4i}}\right) + \frac{i}{3-4i}\left(\frac{\overline{3-4i}}{\overline{3-4i}}\right) \\ &= \frac{2i}{3+4i}\left(\frac{3-4i}{3-4i}\right) + \frac{i}{3-4i}\left(\frac{3+4i}{3+4i}\right) \\ &= \frac{8+6i}{25} + \frac{-4+3i}{25} = \frac{4}{25} + \frac{9}{25}i. \end{aligned}$$

(b) First compute

$$\frac{5+3i}{1+4i} = \frac{5+3i}{1+4i}\left(\frac{1-4i}{1-4i}\right) = \frac{17-17i}{17} = 1-i.$$

Therefore

$$z_2 = \overline{\overline{1-i} + \overline{2i-7}} = \overline{1+i-2i-7} = \overline{-6-i} = -6+i.$$

(c)

$$z_3 = \overline{2i \cdot \overline{2i \cdot \overline{2i}}} = \overline{2i \cdot \overline{2i \cdot -2i}} = \overline{2i \cdot \overline{4}} = \overline{8i} = -8i.$$

(d)

$$z_4 = \frac{1/\sqrt{2}(1+i)}{1/\sqrt{2}(1-i)} = \frac{1+i}{1-i} = \frac{1+i}{1-i} \cdot \left(\frac{1+i}{1+i}\right) = \frac{(1+i)^2}{2} = \frac{2i}{2} = i.$$

Exercise 2.

(1) Two complex numbers are equal if and only if their real and imaginary parts are equal. The real part of $(a+bi)^2$ is a^2-b^2. The imaginary part is $2ab$. Clearly the real part of $c+di$ is c and the imaginary part is d. Therefore $(a+bi)^2 = c+di$ if and only if $a^2-b^2 = c$ and $2ab = d$.

(2) We have to solve for a and b in terms of c and d. Since we are solving the quadratic equation $(a+bi)^2 = c+di$, there will be two solutions for $a+bi$ (although they may be equal).

We consider two cases. First suppose $d = 0$. Then $c + di = c$ is just a real number, and we are trying to solve $(a+bi)^2 = c$. If $c > 0$ then the two solutions are $\pm\sqrt{c}$, ie $a = \pm\sqrt{c}$ and $b = 0$. If $c < 0$ then the two solutions are $\pm i\sqrt{|c|}$, ie $a = 0$ and $b = \pm\sqrt{|c|}$.

Now suppose $d \neq 0$. Then the equation $2ab = d$ yields $b = d/2a$ (note that a cannot be 0 because then d would be 0). Now substituting into the equation $a^2 - b^2 = c$ we get

$$a^2 - \frac{d^2}{4a^2} = c.$$

Multiplying through by a^2 we get

$$a^4 - ca^2 - \frac{d^2}{4} = 0.$$

By the quadratic formula,

$$a^2 = \frac{c \pm \sqrt{c^2 + d^2}}{2}.$$

However, $a^2 > 0$ since a is a real number. In the equation above, since $d^2 > 0$, we have $\sqrt{c^2 + d^2} > c$. Therefore the minus sign above is impossible, ie

$$a^2 = \frac{c + \sqrt{c^2 + d^2}}{2}.$$

Therefore

$$a = \pm\frac{1}{\sqrt{2}}\sqrt{c + \sqrt{c^2 + d^2}}$$

and

$$b = d/2a = \pm\frac{1}{\sqrt{2}}\frac{d}{\sqrt{c + \sqrt{c^2 + d^2}}}.$$

Exercise 3.

(1) When $x_0 = 1, x_2 = 2$ and $x_2 = 3$ we have

$$V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}.$$

(2) Suppose $Vc = y$, where V is the Vandermonde matrix determined by $x_0, x_1, \ldots, x_n \in \mathbb{R}$, and c, y are column vectors with coordinates $c_0, \ldots, c_n$ and $y_0, \ldots y_n$. The equation $Vc = y$ really hides $n + 1$ equations in the variables $c_0, \ldots c_n$, one equation for each row of the system. By the definition of matrix-vector multiplication, the ith row of the system reads

$$x_i^0 c_0 + x_i^1 c_1 + x_i^2 c_2 + \cdots + x_i^n c_n = y_i.$$

Since $x_i^0 = 1$ and $x_i^1 = x_i$, this says exactly that $p(x_i) = y_i$ where $p(x)$ is the polynomial $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$..

Exercise 4.

(a) Note that $i^4 = 1$, so the terms in A_k repeat every 4 terms. Therefore

$$A_k = (i - 1 - i + 1) + (i - 1 - i + 1) + \cdots + i^k.$$

Since the terms in parentheses equal 0, we see that the quantity A_k only depends on the remainder you get when dividing k by 4. Using modular notation, we write

$$A_k = \begin{cases} 0 & k \equiv 0 \mod 4 \\ i & k \equiv 1 \mod 4 \\ i - 1 & k \equiv 2 \mod 4 \\ -1 & k \equiv 3 \mod 4 \end{cases}$$

(For instance, this means $A_k = i$ whenever $k - 1$ is divisible by 4, ie $k = 1, 5, 9, 13, 17, \ldots$.)

(b) Again the terms are periodic every 4 terms so we can write

$$B_k = (i + 1 - i - 1) + (i + 1 - i - 1) + \cdots + i^k.$$

Again the terms in parentheses are 0, so we see that B_k only depends on what k is modulo 4. So we can write

$$B_k = \begin{cases} 0 & k \equiv 0 \mod 4 \\ i & k \equiv 1 \mod 4 \\ i + 1 & k \equiv 2 \mod 4 \\ 1 & k \equiv 3 \mod 4 \end{cases}$$

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(c) This time the terms of C_k are period every other term, so we can write

$$C_k = (i - 1) + (i - 1) + \cdots + i^k.$$

The terms in parentheses are no longer zero, but rather $i - 1$. We can write formulas for C_k depending on the parity of k:

$$C_k = \begin{cases} \frac{k}{2}(i - 1) & k \text{ even} \\ \frac{k-1}{2}(i - 1) + i & k \text{ odd} \end{cases}$$

(d) Let $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, so that $D_k = z^k$. There are two options: one can multiply z^k out by hand to see that $z^8 = 1$. Therefore $D_k = z^k$ only depends on what k is modulo 8; so there are 8 cases that you can write down explicitly.

Alternately, recall from HW 3 that z is the number on the unit circle that is $\pi/4$ radians above the real axis. Note that any complex number which is on the unit circle and is θ radians away from the real axis can be written as $\cos(\theta) + i\sin(\theta)$ (you can see this by using trigonometry!). Therefore $z = \cos(\pi/4) + i\sin(\pi/4)$. Furthermore, the powers of z are obtained by rotating z by $\pi/4$ radians counter-clockwise. More explicitly, z^k is the unit complex number which is $k(\pi/4)$ radians away from the real axis. Therefore

$$D_k = z^k = \cos(k\pi/4) + i\sin(k\pi/4).$$