HW 5 Solns

1)

i) $\begin{vmatrix} 2 & 3 & -1 \\ 0 & 3 & 1 \\ 1 & 2 & -3 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 3 & 1 \end{vmatrix} = 2(-11)+6 = \boxed{-16}$

ii) $\begin{vmatrix} 1 & 5 & 1 \\ 2 & 1 & -1 \\ 4 & 2 & 1 \end{vmatrix} = +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - 5\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$

$= 3 - 30 + 0 = \boxed{-27}$

iii) $\underset{A}{\begin{pmatrix} 2 & 2 & 8 \\ 5 & 2 & 2 \\ 1 & 1 & 4 \end{pmatrix}} \underset{+R_1}{\overset{-2R_3}{\sim}} \underset{B}{\begin{pmatrix} 0 & 0 & 0 \\ 5 & 2 & 2 \\ 1 & 1 & 4 \end{pmatrix}}$ & $\det B = 0$ $\Rightarrow \det A = \boxed{0}$ (type 3 row operations leave determinants unchanged)

iv) $\underset{A'}{\begin{pmatrix} 6 & 1 & 3 \\ 4 & 4 & 2 \\ 10 & -5 & 5 \end{pmatrix}} \overset{-\frac{5}{2}R_2+R_3}{\sim} \underset{B'}{\begin{pmatrix} 6 & 1 & 3 \\ 4 & 4 & 2 \\ 0 & -15 & 0 \end{pmatrix}}$ $\Rightarrow \det(A') = \det(B')$

$= (-1)(-15)\begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix}$

$= 15(12-12) = \boxed{0}$

2)

i) $\underset{A}{\begin{pmatrix} 0 & 0 & 3 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}} \underset{\text{then } R_3 \leftrightarrow R_4}{\overset{R_1 \leftrightarrow R_4}{\sim}} \underset{B}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}}$

$\Rightarrow \det(A) = (-1)^2 \det(B) = \det(B) = 1(-4)(3)(-2)(5) = 5! = \boxed{120}$

(row switches negate the determinant)

B a diagonal matrix

ii) $$\begin{vmatrix} -2 & 3 & 7 & 5 \\ 2 & -4 & -7 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & -5 & 0 \end{vmatrix} = -5 \begin{vmatrix} 2 & -4 & -7 \\ 0 & -1 & -2 \\ 0 & 0 & -5 \end{vmatrix}$$ ← upper triangular

$$= -5(2)(-1)(-5) = \boxed{-50}$$

iii) $$\overset{A}{\begin{pmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & -4 & 0 & 1 & 0 \\ 5 & 0 & -2 & 0 & 1 \\ 0 & 3 & 0 & 2 & 0 \\ -1 & 0 & 2 & 0 & 3 \end{pmatrix}} \underset{R_1+R_4}{\overset{-5R_1+R_3}{\sim}} \begin{pmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & -4 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & -9 \\ 0 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 5 \end{pmatrix}$$

$$\underset{-\frac{1}{3}R_3+R_5}{\overset{\frac{3}{4}R_2+R_4}{\sim}} \underset{B}{\begin{pmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & -4 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & -9 \\ 0 & 0 & 0 & 11/4 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}}$$

so $\det(A) = \det(B) = (1)(-\cancel{4})(3)(11/\cancel{4})(8)$

$= (-33)(8)$

$= \boxed{-264}$

since adding a scalar mult. of one row to another leaves det unchanged

iv) $$\begin{vmatrix} 85 & 17 & 34 & -68 \\ 15 & -5 & 0 & 5 \\ -7 & -14 & 0 & 7 \\ -13 & 13 & 0 & 65 \end{vmatrix} = 34 \begin{vmatrix} 15 & -5 & 5 \\ -7 & -14 & 7 \\ -13 & 13 & 65 \end{vmatrix}$$

$$= 34(5)(7)(13) \begin{vmatrix} 3 & -1 & 1 \\ -1 & -2 & 1 \\ -1 & 1 & 5 \end{vmatrix} = 34(5)(7)(13)\left[3(-10-1)+1(-5+1) +1(-1-2)\right]$$

$$= 34(5)(7)(13)\left[-33-4-3\right]$$

$$= \boxed{34(5)(7)(13)(-40)}$$

Using: $x^{a+b} = x^a x^b$

$(x^a)^b = x^{a \cdot b}$

3)

i) $\left(\left(\left(2^2\right)^2\right)^2\right)^2 \cdot \left(\left(2^2\right)^2\right)^2 \cdot 2^2$

$= \left(\left(2^4\right)^2\right)^2 \cdot \left(2^4\right)^2 \cdot 2^2$

$= \left(2^8\right)^2 \cdot 2^8 \cdot 2^2$

$= 2^{16} \cdot 2^8 \cdot 2^2 = 2^{16+8+2} = 2^{26}$ ✓

And ii) $\left(2 \cdot \left(\left(2 \cdot 2^2\right)^2\right)^2\right)^2 = \left(2 \cdot \left(\left(2^3\right)^2\right)^2\right)^2$

$= \left(2 \cdot \left(2^6\right)^2\right)^2 = \left(2 \cdot 2^{12}\right)^2 = \left(2^{13}\right)^2 = 2^{26}$ ✓

So by ii), $2^{26} = \left(2 \cdot \left((8)^2\right)^2\right)^2 = \left(2 \cdot (64)^2\right)^2$

$= (2 \cdot 4096)^2 = (8192)^2 = \boxed{67108864}$

$5^{13} = 5 \cdot \left(\left(5^3\right)^2\right)^2$

$= 5 \cdot \left((125)^2\right)^2$

$= 5 \cdot (15625)^2$

$= 5 \cdot (244140625)$

$= \boxed{1220703125}$

4)

i) $$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \overset{R_1 \leftrightarrow R_2}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$$\overset{R_1 + R_3}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \overset{R_2 + R_3}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$$\overset{R_3 + R_4}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

ii) $R_4 \Rightarrow 1 \cdot x_4 = 0 \Rightarrow \boxed{x_4 = 0}$

$R_3 \Rightarrow 1 \cdot x_3 = 0 \Rightarrow \boxed{x_3 = 0}$

$R_2 \Rightarrow 1 \cdot x_2 + 1 \cdot \cancel{x_3}^{0} = 1 \Rightarrow \boxed{x_2 = 1}$

$R_1 \Rightarrow 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot \cancel{x_3}^{0} = 0 \Rightarrow x_1 + 1 = 0 \Rightarrow \boxed{x_1 = 1}$

iii) Type II operations involve scaling a row by a nonzero number (in the given field). Over $\mathbb{F}_2$, the only nonzero number is 1, and scaling a row by 1 leaves it unchanged. Thus Type II row operations over $\mathbb{F}_2$ do not change matrices.

Bonus) i) $A' = B^a = (g^b)^a = g^{ba} = g^{ab} = (g^a)^b = A^b = B'$ ✓

ii) $p = 23$, $g = 5$, $a = 4$, $b = 3$. We compute B' ($= A'$ by i)):

$$= g^a = 5^4 = (5^2)^2 = (25)^2 \equiv (2)^2 \equiv 4 \mod 23.$$

So $A' = B' = (g^a)^b = (4)^3 \equiv 16 \cdot 4$

$$\equiv (-7)(4) \mod 23$$

$$\equiv -28 \quad \text{— " —}$$

$$\equiv -5 \quad \text{— " —}$$

$$\equiv \boxed{18} \quad \text{— " —}$$