Math 102 HW 6

Exercise 1.

(1) We want to describe how any permutation $\pi \in \Pi(1,n)$ can be written as a product of transpositions. We'll denote the transposition which switches i and j by $\tau_{i,j}$. If π is the identity transformation then we can write $\pi = \tau_{12}\tau_{12}$ so it can be written as a product of two transpositions (this assumes $n > 1$; when $n = 1$ then $\Pi(1,1)$ only contains the identity element, which could be thought of as the "empty product" of zero transpositions).

Now suppose π is an arbitrary permutation which is not the identity. Some language: if $\pi(i) = i$ we say that π *fixes* the element i. The identity permutation is the only permutation that fixes every element of $\{1,\ldots,n\}$. We can make π closer to the identity by composing with a transposition in the following way. Since π is not the identity, it cannot fix every element, so $\pi(i) \neq i$ for some $i \in \{1,\ldots n\}$. However, if we compose π with the transposition $\tau_{i,\pi(i)}$ then we get a new permutation

$$\sigma = \tau_{i,\pi(i)} \circ \pi$$

with the property that it fixes more points than π! Indeed, if j was fixed by π, then j cannot be i or $\pi(i)$ (it cannot be i because by choice i was not fixed; it cannot be $\pi(i)$ because then $\pi(i) = j = \pi^{-1}(j) = \pi^{-1}(\pi(i)) = i$ but we chose i to not be fixed). Therefore $\sigma(j) = \tau_{i,\pi(i)}\pi(j) = \tau_{i,\pi(i)}(j) = j$. In other words, every element which is fixed by π is also fixed by σ. However, we also have that $\sigma(i) = \tau_{i,\pi(i)}(\pi(i)) = i$, so σ additionally fixes i.

Therefore we can keep modifying our permutation by a transposition in a way that increases the number of fixed elements. Eventually, every element is fixed and we get the identity, so we have an equation of the form

$$\tau_n \circ \cdots \circ \tau_1 \circ \pi = \text{ identity permutation}$$

where the τ_i's are transpositions. Solving for π we get $\pi = \tau_1 \circ \cdots \circ \tau_n$, which is a decomposition of π into transpositions.

(2) Let $\tau_{i,j}$ be the transposition swapping i and j (we may assume $i \neq j$ since otherwise $\tau_{i,j}$ is the identity, which we can write as $\tau_{1,2} \circ \tau_{1,2}$). Let's assume $i < j$. We want to describe how $\tau_{i,j}$ can be expressed as the composition of transpositions of the form $\tau_{k,k+1}$ (which we call *simple* transpositions). We do it in two stages. First, since $\tau_{i,j}(i) = j$ we just try to create a permutation π out of simple transpositions such that $\pi(i) = j$. This is not too hard: we can take

$$\pi = \tau_{j-1,j} \circ \tau_{j-2,j-1} \circ \cdots \circ \tau_{i,i+1}.$$

From this definition we see that $\pi(i) = j$ (since the first transposition takes i to $i+1$ and the next takes $i+1$ to $i+2$ etc.) This permutation fixes every element less than i and greater than j. For the numbers in between we have $\pi(k) = k-1$ (except i, for which we have $\pi(i) = j$. In particular $\pi(j) = j-1$, but our goal is to send j to i. Therefore we consider the permutation

$$\sigma = \tau_{i,i+1} \circ \ldots \tau_{j-2,j-1} \circ \pi = \tau_{i,i+1} \circ \ldots \tau_{j-2,j-1} \circ \tau_{j-1,j} \circ \tau_{j-2,\mathrm{j}-1} \circ \cdots \circ \tau_{i,i+1}.$$

We claim that $\sigma = \tau_{i,j}$. Indeed, σ takes j to i and i to j. Furthermore, σ fixes every element less than i and greater than j. For the elements in between $i < k < j$ we have

$$\sigma(k) = \tau_{i,i+1} \circ \ldots \tau_{j-2,j-1}(\pi(k)) = \tau_{i,i+1} \circ \ldots \tau_{j-2,j-1}(k-1) = k$$

so $\sigma(k) = k$. Therefore σ switches i and j, and fixes every other element, so σ is the transposition $\tau_{i,j}$.

Exercise 2.

(1) We follow the procedure of the first part, namely we multiply π by transposition such that the number of fixed points increases. So first take some point that is not fixed, for instance $\pi(1) = 2$. Then

$$\pi_1 = \tau_{1,2} \circ \pi$$

fixes 1. However $\pi_1(2) = \tau_{1,2}(\pi(2)) = \tau_{1,2}(4) = 4$, so 2 is not fixed by π_1. We can fix this by taking

$$\pi_2 = \tau_{2,4} \circ \pi_1 = \tau_{2,4} \circ \tau_{1,2} \circ \pi.$$

So π_2 fixes both 1 and 2. Now $\pi_2(3) = 6$ so we let

$$\pi_3 = \tau_{3,6} \circ \pi_2 = \tau_{3,6}\tau_{2,4}\tau_{1,2}\pi.$$

which fixes $1, 2, 3$. (We're using the shorthand $\sigma_1\sigma_2 = \sigma_1 \circ \sigma_2$.) Now $\pi_3(4) = 6$ so let

$$\pi_4 = \tau_{4,6}\pi_3 = \tau_{4,6}\tau_{3,6}\tau_{2,4}\tau_{1,2}\pi.$$

which fixes $1, 2, 3, 4$. Next, $\pi_4(5) = 6$ so we let

$$\pi_5 = \tau_{5,6}\pi_4 = \tau_{5,6}\tau_{4,6}\tau_{3,6}\tau_{2,4}\tau_{1,2}\pi.$$

Now $\pi_5(6) = 7$ so we let

$$\pi_6 = \tau_{6,7}\pi_5 = \tau_{6,7}\tau_{5,6}\tau_{4,6}\tau_{3,6}\tau_{2,4}\tau_{1,2}\pi. \tag{1}$$

This permutation fixes $1, 2, 3, 4, 5, 6$ so it must also fix 7 and so $\pi_6 =$ identity . Substituting for π and solving we get

$$\pi = \tau_{1,2}\tau_{2,4}\tau_{3,6}\tau_{4,6}\tau_{5,6}\tau_{6,7}.$$

(2) Since π_6 is the identity, Equation 1 states that

$$(\tau_{6,7}\tau_{5,6}\tau_{4,6}\tau_{3,6}\tau_{2,4}\tau_{1,2})\,\pi = \text{identity}$$

which means

$$\pi^{-1} = \tau_{6,7}\tau_{5,6}\tau_{4,6}\tau_{3,6}\tau_{2,4}\tau_{1,2}.$$

We have that

$$(\pi^{-1}(1), \pi^{-1}(2), \pi^{-1}(3), \pi^{-1}(4), \pi^{-1}(5), \pi^{-1}(6), \pi^{-1}(7)) = (5, 1, 4, 2, 7, 3, 6).$$

Exercise 3.

(1) We can compute the determinant of A_λ using the usual formula for 2×2 determinants:

$$\det A_\lambda = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc.$$

By the quadratic formula, $\det A_\lambda = 0$ if and only if

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

(2) If $\det A_\lambda = 0$ then from part (1), $\lambda^2 - (a+d)\lambda + ad - bc = 0$. We can rewrite this as

$$\lambda d - ad + bc = \lambda^2 - \lambda a = \lambda(\lambda - a).$$

Now we compute

$$\begin{aligned} A_0 v &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} b \\ \lambda - a \end{pmatrix} \\ &= \begin{pmatrix} ab + b(\lambda - a) \\ cb + d(\lambda - a) \end{pmatrix} = \begin{pmatrix} \lambda b \\ \lambda d - ad + bc \end{pmatrix} \\ &= \begin{pmatrix} \lambda b \\ \lambda(\lambda - a) \end{pmatrix} \end{aligned}$$

where the last equality is from the previous equation. This shows that $A_0 v = \lambda v$.

Exercise 4. Consider the $2n \times 2n$ matrix $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where A, B, C are $n \times n$ real matrices and 0 is the $n \times n$ zero matrix. The formula for the determinant using permutations is

$$\det(M) = \sum_{\pi \in \Pi(1,2n)} \operatorname{sign}(\pi) \prod_{i=1}^{2n} m_{i,\pi(i)}$$

where $\operatorname{sign}(\pi) \in \{1, -1\}$ denotes the sign of the permutation π and $m_{i,j}$ denotes the i, j-the entry of M. However, $m_{i,j} = 0$ whenever $i > n$ and $j \leq n$ (since then $m_{i,j}$ is an entry in the lower left corner of M which is the 0 matrix). Therefore any permutation which sends some $i > n$ to something which is $\leq n$ contributes 0 to the sum above. The permutations which do not automatically contribute 0 are all the permutations which send $\{n-1, \ldots, 2n\}$ to $\{n-1, \ldots, 2n\}$ (ie if $i > n$ then $\pi(i)$ is also $> n$). Note that this is the same as saying the permutations sends $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$. Thus any permutation π not contributing 0 can be expressed using two permutations $\pi_1, \pi_2 \in \{1, \ldots, n\}$ by

$$\pi(i) = \begin{cases} \pi_1(i) & i \leq n \\ \pi_2(i-n) + n & i > n \end{cases}.$$

Note that $\operatorname{sign}(\pi) = \operatorname{sign}(\pi_1)\operatorname{sign}(\pi_2)$, since if π_1 can be written as a product of r transpositions and π_2 can be written as a product of s transpositions, then π

can be written as a product of $r+s$ transpositions (and now you can check the different cases of r, s even and odd). We can write the determinant of M as

$$\begin{aligned}
\det(M) &= \sum_{\pi_1,\pi_2\in\Pi(1,n)} \mathrm{sign}(\pi_1)\mathrm{sign}(\pi_2)\prod_{i=1}^{2n} m_{i,\pi(i)} = \\
&= \sum_{\pi_1,\pi_2\in\Pi(1,n)} \mathrm{sign}(\pi_1)\mathrm{sign}(\pi_2)\prod_{i=1}^{n} m_{i,\pi_1(i)}\prod_{i=1}^{n} m_{i+n,\pi_2(i)+n} \\
&= \sum_{\pi_1,\pi_2\in\Pi(1,n)} \mathrm{sign}(\pi_1)\mathrm{sign}(\pi_2)\prod_{i=1}^{n} a_{i,\pi_1(i)}\prod_{i=1}^{n} c_{i,\pi_2(i)} \\
&= \left(\sum_{\pi_1\in\Pi(1,n)} \mathrm{sign}(\pi_1)\prod_{i=1}^{n} a_{i,\pi_1(i)}\right)\left(\sum_{\pi_2\in\Pi(1,n)} \mathrm{sign}(\pi_2)\prod_{i=1}^{n} c_{i,\pi_2(i)}\right) = \det(A)\det(C).
\end{aligned}$$