MATH 102 – HOMEWORK ASSIGNMENT 7

Due Friday, November 17th, 2017 before the lecture.
Handwritten submissions only.

Exercise 1 (4 points)**.**
We conduct Gaussian elimination and the LU decomposition over the field of complex numbers.

(1) Bring the following 2×2 system $A \cdot z = b$ of linear equations over the field $\mathbb{C}$ into triangular form:

$$\begin{pmatrix} 2+\mathbf{i} & 3\mathbf{i} \\ -1-\mathbf{i} & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1+\mathbf{i} \\ 1-\mathbf{i} \end{pmatrix}.$$

(2) Compute the solution $z = (z_1, z_2) \in \mathbb{C}^2$ of the system above.
(3) Determine a lower triangular matrix $L' \in \mathbb{C}^{2\times 2}$ such that $L'A$ is upper triangular.
(4) Determine a lower triangular matrix $L \in \mathbb{C}^{2\times 2}$ and an upper triangular matrix $U \in \mathbb{C}^{2\times 2}$ such that $A = LU$.

Solution 1. (1) We multiply the first row of the system by $(-1-\mathbf{i})/(2+\mathbf{i})$ and the subtract the resulting row from the second row.

$$\left(\begin{array}{cc|c} 2+\mathbf{i} & 3\mathbf{i} & 1+\mathbf{i} \\ -1-\mathbf{i} & 2 & 1-\mathbf{i} \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} -1-\mathbf{i} & 3\mathbf{i}\frac{-1-\mathbf{i}}{2+\mathbf{i}} & (1+\mathbf{i})\frac{-1-\mathbf{i}}{2+\mathbf{i}} \\ -1-\mathbf{i} & 2 & 1-\mathbf{i} \end{array}\right) = \left(\begin{array}{cc|c} -1-\mathbf{i} & \frac{3-3\mathbf{i}}{2+\mathbf{i}} & \frac{-2\mathbf{i}}{2+\mathbf{i}} \\ -1-\mathbf{i} & 2 & 1-\mathbf{i} \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} -1-\mathbf{i} & \frac{3-3\mathbf{i}}{2+\mathbf{i}} & \frac{-2\mathbf{i}}{2+\mathbf{i}} \\ 0 & 2-\frac{3-3\mathbf{i}}{2+\mathbf{i}} & 1-\mathbf{i}-\frac{-2\mathbf{i}}{2+\mathbf{i}} \end{array}\right) = \left(\begin{array}{cc|c} -1-\mathbf{i} & \frac{3-3\mathbf{i}}{2+\mathbf{i}} & \frac{-2\mathbf{i}}{2+\mathbf{i}} \\ 0 & \frac{1+5\mathbf{i}}{2+\mathbf{i}} & \frac{2\mathbf{i}+(1-\mathbf{i})(2+\mathbf{i})}{2+\mathbf{i}} \end{array}\right) = \left(\begin{array}{cc|c} -1-\mathbf{i} & \frac{3-3\mathbf{i}}{2+\mathbf{i}} & \frac{-2\mathbf{i}}{2+\mathbf{i}} \\ 0 & \frac{1+5\mathbf{i}}{2+\mathbf{i}} & \frac{3+\mathbf{i}}{2+\mathbf{i}} \end{array}\right)$$

(2) The second component satisfies

$$z_2 = \frac{3+\mathbf{i}}{2+\mathbf{i}} \div \frac{1+5\mathbf{i}}{2+\mathbf{i}} = \frac{3+\mathbf{i}}{1+5\mathbf{i}}$$

The first component satisfies

$$\begin{aligned} & (-1-\mathbf{i})z_1 + \frac{3-3\mathbf{i}}{2+\mathbf{i}} z_2 = \frac{-2\mathbf{i}}{2+\mathbf{i}} \\ \iff \quad & (2+\mathbf{i})(-1-\mathbf{i})z_1 + (3-3\mathbf{i})z_2 = -2\mathbf{i} \\ \iff \quad & (2+\mathbf{i})(-1-\mathbf{i})z_1 + (3-3\mathbf{i})\frac{3+\mathbf{i}}{1+5\mathbf{i}} = -2\mathbf{i} \\ \iff \quad & (2+\mathbf{i})(-1-\mathbf{i})z_1 = \frac{-2\mathbf{i}(1+5\mathbf{i}) - (3-3\mathbf{i})(3+\mathbf{i})}{(1+5\mathbf{i})} \end{aligned}$$

$$\iff \quad z_1 = \frac{-2\mathbf{i}(1+5\mathbf{i}) - (3-3\mathbf{i})(3+\mathbf{i})}{(2+\mathbf{i})(-1-\mathbf{i})(1+5\mathbf{i})} = \frac{2\mathbf{i}(1+5\mathbf{i}) + (3-3\mathbf{i})(3+\mathbf{i})}{(2+\mathbf{i})(1+\mathbf{i})(1+5\mathbf{i})} = \frac{2\mathbf{i}-10+12-6\mathbf{i}}{(1+3\mathbf{i})(1+5\mathbf{i})} = \frac{2-4\mathbf{i}}{-15+8\mathbf{i}}$$

(3) Keeping track of the elementary row operations, we identify L' as

$$L' = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} (-1-\mathbf{i})/(2+\mathbf{i}) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (-1-\mathbf{i})/(2+\mathbf{i}) & 0 \\ (1+\mathbf{i})/(2+\mathbf{i}) & 1 \end{pmatrix}$$

(4) The above lower triangular matrix L' satisfies $L'A = U$, where U is the upper triangular form derived initially. Hence $A = LU$, where $L = (L')^{-1}$. Explicitly,

$$\begin{aligned} L &= \begin{pmatrix} (-1-\mathbf{i})/(2+\mathbf{i}) & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} (2+\mathbf{i})/(-1-\mathbf{i}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} (2+\mathbf{i})/(-1-\mathbf{i}) & 0 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

Exercise 2 (4 points).
Let $A \in \mathbb{R}^{n\times n}$ be an upper triangular matrix. Show that the determinant of A is the product of the diagonal entries, i.e.,

$$\det(A) = \prod_{i=1}^{n} a_{ii}.$$

Hint: Use induction.

Solution 2.
The statement is clearly true for $n = 1$. Now assume the statement is true for some number $n \in \mathbb{N}$. If $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is upper triangular, then the Laplace expansion along the first row gives

$$\det(A) = a_{11} \det(M_{11}),$$

where M_{11} is the $n \times n$ matrix obtained from A after deleting the first row and first column. Here we have used that only the first entry in the first row of A is possibly non-zero. Since we assume the statement to be true for the case n, we get that

$$\det(A) = a_{11} \det(M_{11}) = a_1 1 \prod_{k=2}^{n+1} = \prod_{k=1}^{n+1} a_{kk}$$

By the principle of induction, the claim follows.

Exercise 3 (4 points).
Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and let $\pi \in \Pi(1, n)$ be a permutation. We let $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ be the $n \times n$ matrix whose i-th row is the $\pi(i)$-th row of A. This means that

$$b_{ij} = a_{\pi(i),j}$$

for $1 \leq i, j \leq n$. Show that

$$\det(B) = \operatorname{sgn}(\pi) \cdot \det(A).$$

Hint: use the Leibniz formula.

Solution 3.
Let A, B, and π as the statement of the problem. The Leibniz formula then gives

$$\det(B) = \sum_{\sigma\in\Pi(1,n)} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)}.$$

By the definition of B, this is just

$$\det(B) = \sum_{\sigma\in\Pi(1,n)} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\pi(i),\sigma(i)}.$$

Since the factors in the product can be reordered, we may reorder the factors by the permutation π^{-1}, which gives

$$\det(B) = \sum_{\sigma\in\Pi(1,n)} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\pi(\pi^{-1}(i)),\sigma(\pi^{-1}(i))} = \sum_{\sigma\in\Pi(1,n)} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(\pi^{-1}(i))}.$$

Since $\sigma = \sigma \circ \pi^{-1} \circ \pi$, we also have $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma \circ \pi^{-1}) \cdot \operatorname{sgn}(\pi)$, and hence

$$\det(B) = \sum_{\sigma\in\Pi(1,n)} \operatorname{sgn}(\sigma \circ \pi^{-1}) \cdot \operatorname{sgn}(\pi) \prod_{i=1}^{n} a_{i,\sigma(\pi^{-1}(i))} = \operatorname{sgn}(\pi) \sum_{\sigma\in\Pi(1,n)} \operatorname{sgn}(\sigma \circ \pi^{-1}) \prod_{i=1}^{n} a_{i,\sigma(\pi^{-1}(i))}.$$

Now, in the sum, instead of going over all permutations $\sigma \in \Pi(1,n)$ and composing them with π^{-1}, we just go over all permutations directly:

$$\sum_{\sigma\in\Pi(1,n)} \operatorname{sgn}(\sigma\circ\pi^{-1}) \prod_{i=1}^{n} a_{i,\sigma(\pi^{-1}(i))} = \sum_{\sigma\in\Pi(1,n)} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

The reason is that composing all members of $\Pi(1,n)$ gives again all members of $\Pi(1,n)$. The claim now follows.

Exercise 4 (1+1+2 points)**.**
For each $\pi \in \Pi(1,n)$ we define the permutation matrix $P_\pi = (p_{ij}) \in \mathbb{R}^{n\times n}$ by the property

$$p_{ij} = \begin{cases} 1 & \text{if } \pi(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

(1) Show that for every n-dimensional vector with entries $v_1, \ldots, v_n \in \mathbb{R}$ we have

$$P_\pi \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_{\pi(1)} \\ v_{\pi(2)} \\ \vdots \\ v_{\pi(n)} \end{pmatrix}$$

(2) Show that $\det(P_\pi) = \operatorname{sgn}(\pi)$.
(3) Show that for $\pi, \rho \in \Pi(1,n)$ we have

$$P_\rho \circ P_\pi = P_{\pi\circ\rho}$$

Solution 4. (1) Let $v \in \mathbb{R}^n$ and $\pi \in \Pi(1,n)$. Write

$$w = P_\pi \cdot v.$$

The i-th coordinate of w is

$$w_i = p_{i1}v_1 + \cdots + p_{in}v_n = p_{i,\pi(i)}v_{\pi(i)},$$

where we have used the definition of the entries.

(2) Now, whenever $\pi \in \Pi(1,n)$, we observe that

$$\det(P_\pi) = \sum_{\sigma\in P(1,n)} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} p_{i,\sigma(i)}$$

Let $\sigma \in P(1,n)$ with $\sigma \neq \pi$. Then there is $1 \le i \le n$ with $\sigma(i) \neq \pi(i)$. But then $p_{i,\sigma(i)} = 0$. Consequently, the only term that remains in the Leibniz formula is the summand associated to $\sigma = \pi$. Thus

$$\det(P_\pi) = \operatorname{sgn}(\pi) \prod_{i=1}^{n} p_{i,\pi(i)} = \operatorname{sgn}(\pi).$$

(3) To prove the last item, let $\pi, \rho \in \Pi(1,n)$. Let c_{ij} be the entry of the matrix–matrix product $P_\rho P_\sigma$, and let p^ρ_{ij} and p^σ_{ij} denote the entries of the latter two matrices. Using the definition of the matrix–matrix product, the definition of P_ρ, and then the definition of P_π, we find that

$$c_{ij} = \sum_{k=1}^{n} p^\rho_{i,k} p^\sigma_{k,j} = p^\rho_{i,\rho(i)} p^\sigma_{\rho(i),j} = \begin{cases} 1 & \text{if } \pi(\rho(i)) = j, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, these are just the entries of the permutation matrix associated to $\pi \circ \rho$.