MATH 102 – HOMEWORK ASSIGNMENT 8

Due Friday, November 24th, 2017 before the lecture.
Handwritten submissions only.

Exercise 1 (4 points)**.**
Let $A, B \in \mathbb{R}^n$. Prove the following:

(1) $\det(A^k) = \det(A)^k$ for $k \in \mathbb{N}$.
(2) $\det(\alpha A) = \alpha^n \det(A)$
(3) $\det(-A) = (-1)^n \det(A)$.
(4) $\det(AB) = \det(BA)$

Solution 1.

(1) The statement is true for $k = 1$. Now suppose it is true for some $k \in \mathbb{N}$. Then we see that

$$\det(A^{k+1}) = \det(A^k)\det(A) = \det(A)^k \det(A) = \det(A)^{k+1}.$$

Hence the statement is true for $k + 1$. The claim now follows by the principle of induction.

(2) We have

$$\det(\alpha A) = \sum_{\sigma \in \Pi(1,n)} \operatorname{sgn}(\sigma) \prod_{l=1}^{n} \alpha a_{i,\sigma(i)} = \alpha^n \sum_{\sigma \in \Pi(1,n)} \operatorname{sgn}(\sigma) \prod_{l=1}^{n} a_{i,\sigma(i)} = \alpha^n \det(A).$$

(3) This follows from the previous item when choosing $\alpha = -1$.

(4) We have

$$\det(AB) = \det(A)\det(B) = \det(BA).$$

Exercise 2 (4 points)**.**
Let $A \in \mathbb{R}^n$ be invertible with $n > 2$. Prove that

$$\det(\operatorname{adj} A) = (\det(A))^{n-1}$$

Solution 2.
Since

$$A \cdot \operatorname{adj} A = \det(A) \cdot \mathrm{Id}_n$$

it follows that

$$\det(A \cdot \operatorname{adj} A) = \det(\det(A) \cdot \mathrm{Id}_n)$$

and hence

$$\det(A) \cdot \det(\operatorname{adj} A) = \det(A)^n \det(\mathrm{Id}_n) = \det(A)^n.$$

Since A is invertible, we have $\det(A) \neq 0$. The claim follows.

Exercise 3 (4 points)**.**
Compute the following determinant:

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ e & 0 & e^{\pi} & 4 & 5 & 1 & \sqrt{\pi} \\ e^2 & 1 & \frac{17}{31} & \sqrt{6} & \sqrt{7} & \sqrt{8} & \sqrt{10} \\ e^3 & 0 & -e & \pi & e & 0 & \pi^e \\ e^4 & 0 & 10001 & 0 & \pi^{-1} & 0 & e^2\pi \\ e^6 & 0 & \sqrt{2} & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution 3.
Two rows are the same, so the determinant is zero.

Exercise 4 (4 points)**.**
Let W, X, Y, Z be vector spaces. Let $A \in L(W, X)$ and $B \in L(Y, Z)$. Consider the mapping

$$T : L(X, Y) \to L(W, Z), \quad C \mapsto B \circ C \circ A$$

Explain the mapping T in your own words (where does it map from, where does it map to, what does it do?) and show show that

$$T \in L\left(L(W, X), L(Y, Z)\right).$$

Solution 4.
The mapping T takes a linear mapping from X to Y and maps it to a linear mapping from W to Z. If $L \in L(X, Y)$ is a linear mapping from X to Y, then $T(L)$ is the composition $B \circ L \circ A$, which maps from W to Z:

$$W \xrightarrow{\ A\ } X \xrightarrow{\ L\ } Y \xrightarrow{\ B\ } Z.$$

We know that composition from the right is linear. Hence we have a linear mapping

$$T_1 : L(X, Y) \to L(W, Y), \quad L \mapsto L \circ A.$$

We know that composition from the left is linear. Hence we have a linear mapping

$$T_2 : L(W, Y) \to L(W, Z), \quad L \mapsto B \circ L.$$

It is easily seen that $T = T_2 \circ T_1$, because for every $L \in L(X, Y)$ we have

$$(T_2 \circ T_1)(L) = T_2 (L \circ A) = B \circ L \circ A = T(L).$$

Since T is the composition of linear mappings, T is linear.