MATH 102 – HOMEWORK ASSIGNMENT 9

Due December 8th, 2017 before the lecture.
Handwritten submissions only.

Exercise 1 (6 points)**.**
Let $v_1, \ldots, v_m \in \mathbb{R}^n$ be vectors and let $w_1, \ldots, w_m$ be obtained by one of the following three procedures:

- Switching vectors:

$$w_1 = v_1, \ \ldots, w_{i-1} = v_{i-1}, w_i = v_j, w_{i+1} = v_{i+1}, \ldots, w_{j-1} = v_{j-1}, w_j = v_i, w_{j+1} = v_{j+1}, \ldots, w_m = v_m$$

- Scaling the i-th vector by a nonzero factor $\alpha \in \mathbb{R}$:

$$w_1 = v_1, \ldots, w_{i-1} = v_{i-1}, w_i = \alpha v_i, w_{i+1} = v_{i+1}, \ldots, w_m = v_m.$$

- Adding the α-th multiple of v_j to the i-th vector:

$$w_1 = v_1, \ldots, w_{j-1} = v_{j-1}, w_j = v_j + \alpha v_i, w_{j+1} = v_{j+1}, \ldots w_m = v_m.$$

For each of these three constructions, show that $w_1, \ldots, w_m$ has the same span as $v_1, \ldots, v_m$ and show that $w_1, \ldots, w_m$ is linearly independent if and only if $v_1, \ldots, v_m$ is.

Solution 1.
Let $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ and write

$$v = \alpha_1 v_1 + \cdots + \alpha_m v_m. \tag{1}$$

- If $w_1, \ldots, w_n$ is constructed by switching the i-th and the j-th vector, then

$$v = \alpha_1 w_1 + \ldots + \alpha_{i-1} w_{i-1} + \alpha_i w_i + \alpha_{i+1} v_{i+1} + \ldots + \alpha_{j-1} w_{j-1} + \alpha_i w_j + \alpha_{j+1} w_{j+1} + \ldots + \alpha_m w_m. \tag{2}$$

- If $w_1, \ldots, w_n$ is constructed by scaling the i-th vector v_i with $\alpha \in \mathbb{R}$ non-zero, and leaving the other vectors the same, then

$$v = \alpha_i w_1 + \ldots + \alpha_{i-1} w_{i-1} + \frac{\alpha_i}{\alpha} w_i + \alpha_{i+1} w_{i+1} + \ldots + \alpha_m. \tag{3}$$

- Adding the α-th multiple of v_j to the i-th vector:

$$\begin{aligned} v = \alpha_1 v_1 + \ldots + \alpha_{j-1} w_{j-1} + \alpha_j w_j + \alpha_{j+1} w_{j+1} + \ldots \\ \ldots + \alpha_{i-1} w_{i-1} + (\alpha_i - \alpha\alpha_j) w_i + \alpha_{j+1} w_{i+1} + \ldots + \alpha_m w_m. \end{aligned} \tag{4}$$

Hence, v is in the span of the vectors $w_1, \ldots, w_m$ if v is the span of the vectors $v_1, \ldots, v_m$. The coefficients in the linear combination (2)/(3)/(4) are all zero if the coefficients in the linear combination (1) are all zero; hence the $w_1, \ldots, w_m$ are linearly independent if the $v_1, \ldots, v_m$ are linearly independent.

Lastly, whenever $w_1, \ldots, w_m$ is obtained from $v_1, \ldots, v_m$ by the first/second/third operation, then $v_1, \ldots, v_m$ is obtained from $w_1, \ldots, w_m$ by an analogous first/second/third operation. As a consequence, the above implications are in fact equivalences.

Exercise 2 (2 points)**.**
Let V be a vector space and let $v_1, \ldots, v_m \in V$ be vectors. Show that if they are linearly dependent if and only if there exists $1 \leq i \leq m$ such that v_i is a linear combination of $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m$.

Solution 2.
First, assume that v_i is a linear combination of $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m$.

So there exist $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_m \in \mathbb{R}$ with

$$v_i = \alpha_1 v_1 + \cdots + \alpha_{i-1} v_{i-1} + \alpha_{i+1} v_{i+1} + \cdots + \alpha_m v_m.$$

This implies

$$0 = \alpha_1 v_1 + \cdots + \alpha_{i-1} v_{i-1} + (-1) v_i + \alpha_{i+1} v_{i+1} + \cdots + \alpha_m v_m.$$

Since at least one coefficient in that linear combination is non-zero, we infer linear dependence.

On the other hand, if $v_1, \dots, v_m$ are linearly dependent, there exist $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ not all zero such that

$$0 = \alpha_1 v_1 + \cdots + \alpha_m v_m.$$

Since the coefficients are not all zero, let's pick a non-zero coefficient, say, $\alpha_i \neq 0$. Then

$$\begin{aligned} & 0 = \alpha_1 v_1 + \cdots + \alpha_{i-1} v_{i-1} + \alpha_i v_i + \alpha_{i+1} v_{i+1} + \cdots + \alpha_m v_m \\ \iff \quad & -\alpha_i v_i = \alpha_1 v_1 + \cdots + \alpha_{i-1} v_{i-1} + \alpha_{i+1} v_{i+1} + \cdots + \alpha_m v_m \\ \iff \quad & v_i = \frac{\alpha_1}{-\alpha_1} v_1 + \cdots + \frac{\alpha_{i-1}}{-\alpha_1} v_{i-1} + \frac{\alpha_{i+1}}{-\alpha_1} v_{i+1} + \cdots + \frac{\alpha_m}{-\alpha_1} v_m. \end{aligned}$$

This had to be shown.

Exercise 3 (6 points)**.**
Find bases for the kernels and the ranges of the following matrices.

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 2 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -4 & 2 & 0 & 0 & 0 \\ 5 & -1 & 1 & 3 & 7 \\ -4 & 2 & 0 & 8 & 0 \\ 13 & 0 & 0 & 0 & 0 \\ 1 & -1 & 4 & 0 & 0 \end{pmatrix}.$$

Solution 3.
Bases are given as follows:

$$\ker A = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix} \right\}, \quad \operatorname{range} A = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 2 \\ -5 \end{pmatrix} \right\},$$

$$\ker B = \operatorname{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}, \quad \operatorname{range} B = \operatorname{span} \left\{ \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

The matrix C is invertible. Hence the columns are linearly independent and span its range, which is $\mathbb{R}^5$. Since C is invertible, the kernel of C is the vector space that contains only the zero vector. The empty set is a basis for $\ker C = \{0\}$.

Exercise 4 (2 points)**.**
Assume that $v_1, \dots, v_n \in \mathbb{R}^n$ and $w_1, \dots, w_n \in \mathbb{R}^n$ are two different bases of $\mathbb{R}^n$, and that they are also the columns of the matrices $A, B \in \mathbb{R}^{n \times n}$:

$$A = \left(\begin{array}{c|c|c|c} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{array} \right), \quad B = \left(\begin{array}{c|c|c|c} | & | & & | \\ w_1 & w_2 & \dots & w_n \\ | & | & & | \end{array} \right).$$

Prove that the basis transformation from the basis $v_1, \dots, v_n$ to the basis $w_1, \dots, w_n$ is given by the matrix $B^{-1}A$. In other words, whenever $x \in \mathbb{R}^n$ is a vector with the basis representations

$$x = \alpha_1 v_1 + \cdots + \alpha_n v_n = \beta_1 w_1 + \cdots + \beta_n w_n,$$

then

$$B^{-1} A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}.$$

Solution 4.
The matrix A is invertible. The reason is that A has n columns that form a basis of $\mathbb{R}^n$, so the rank of A is n. As a consequence, A is invertible. Similarly, we find that the matrix B is invertible.

We have $x = \alpha_1 v_1 + \cdots + \alpha_n v_n$ if and only if

$$A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = v.$$

Similarly, we have $x = \beta_1 w_1 + \cdots + \beta_n w_n$ if and only if

$$B \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = v.$$

Putting everything together, we find

$$A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = B \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}.$$

Applying the matrix B^{-1} to both sides gives the desired result.